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FIELD OF VIEW IN THE OCLINOMETER[1]

By L. E. Dodd

ABSTRACT

The paper presents two phases of the subject. (I) Schematic description of an apparently new type of optical instrument, designed by Liggett and Dodd, which permits conditions within wide limits for accommodation of the eyes independent of convergence, and also the measurement of muscular imbalance (strabismus) and range of fusion. It uses plane mirrors rather than refracting prisms, thus eliminating distortion, chromatic aberration and astigmatism in the images. The optical system is essentially different from that of the short-base range finder and of the Helmholtz stereoscope, in that (a) rays enter the eyes from the *outer* rather than the inner mirrors, and (b) the outer mirrors are variable in position through large angles. (II) Derivation of geometrical expressions for (1) magnitude of "field of view" (that is, angular range of visibility of secondary image of stationary point-object), (2) direction of field of view, and (3) boundaries of useful areas of mirrors, —applicable to a pair of independently variable plane mirrors with parallel axes in the respective planes, as adapted in an instrument of this kind (one pair of mirrors to each eye).

The designing entailed the fulfilling of several requirements having somewhat narrow limits and culminated in finding a suitable combination of some half-dozen variable quantities.

Introduction

Any device using refracting prisms of very appreciable strength for the measurement or correction of muscular imbalance (strabismus) and eye strain, or for measuring the range of fusion, is subject to distortion, chromatic aberration and astigmatism in the images. The alternative seems to be some arrangement of mirrors.

A study of the possibility of such a device for practical use at different object distances, and for both internal and external strabismus, to

[1] Paper appearing on the program of the American Physical Society, Berkeley meeting, June 21, 22, 1929, with the title, "Field of View of Secondary Image, with Point-Object Stationary, for Two Independently Variable Plane Mirrors Having Limited Widths." The words "with point-object stationary" refer to the object distance as a variable parameter

be constructed of plane mirrors rather than refracting prisms, was proposed to the writer in the fall of 1927 by Dr. J. R. Liggett, of the Department of Psychology, U. C. L. A., who has collaborated in the design. The result of this study, an instrument called the oclinometer, is outlined schematically in this paper.

Optical instruments like the stereoscope of Helmholtz, or the optically similar short-base range finder, in which beams of light are deflected by plane reflecting surfaces, *employ the two inside mirrors for the secondary reflection*. These mirrors are fixed symmetrically in position with respect to the longitudinal axis (axis of symmetry) of the system, at an angle of 90° with each other, and from them the rays enter the eyes.

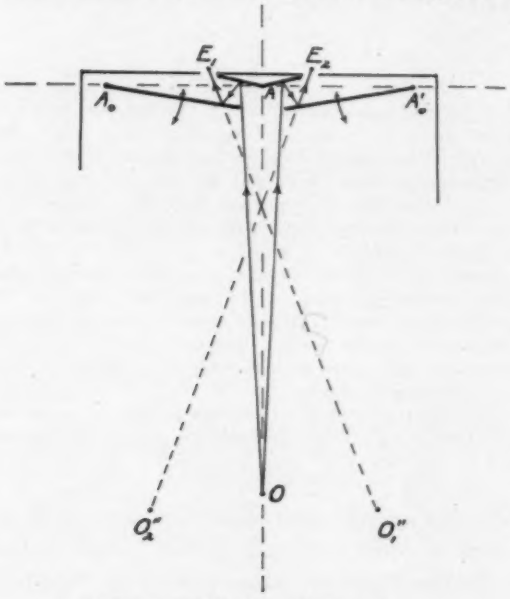

FIG. 1. *Optical system of oclinometer.*

THE OCLINOMETER

It is practicable to reverse the light rays in a system of this kind for adaptation to muscular imbalance and in general to permit convergence of the eyes independent of their accommodation. The oclinometer is an instrument of this apparently new type. (Detailed description of its construction is unnecessary here and will be presented separately.) It uses a system of plane mirrors, Fig. 1, of the same general character as

that of the range finder, but so designed that *the eyes look into the outer mirrors*. These are variable about parallel vertical axes, A_0, A_0'. The inner mirrors are fixed symmetrically with respect to the longitudinal axis at an angle of 140° with each other. This scheme requires that the two inner mirrors be limited to a combined projected width of somewhat less than the interocular distance. (This distance in the instrument could be made adjustable by a simple screw device as in Helmholtz's modification of Brewster's stereoscope. See *Physiological Optics*, English Translation, Vol. III, p. 350.)

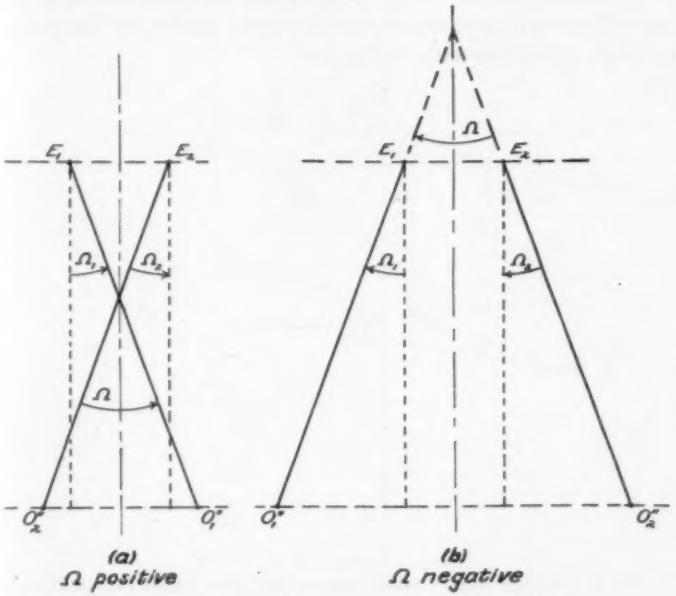

FIG. 2. *Inter-image angle,* $\Omega=\Omega_1+\Omega_2$, *corresponding to cases of internal and external strabismus.*

REQUIREMENTS TO BE FULFILLED IN THE INSTRUMENT

In arriving at this design, variable (including parametric) quantities dealt with were: (1) widths of inner mirrors; (2) widths of outer mirrors; (3) distance from common axis of inner mirrors to the axis of either of the outer mirrors; (4) distance from mirrors to source, supposed placed on the longitudinal axis of the system; (5) eye positions (centers of rotation); (6) angular positions of the mirrors, two of which, the inner ones, are fixed; (7) relative positions of the mirror axes.

Then choice of dimensions in the system had to be made such that

(a) for all distances of the source from the mirrors, the two secondary images are maintained visible both at the same time at a sufficiently large maximum separation (better than 40° was attained in the present design), also (b) this maximum inter-image angle (taken here as $\Omega = \Omega_1 + \Omega_2$, Fig. 2) may have either sign in order to be adaptable to eyes turned abnormally either outward or inward, and further, (c) the design would be for a sufficiently compact optical instrument.

METHOD OF ANALYSIS

The symmetry of the system permits the analysis to be made with *one pair of mirrors*, an inner one and its outer mirror, on the same side of the longitudinal axis with it, Fig. 1.

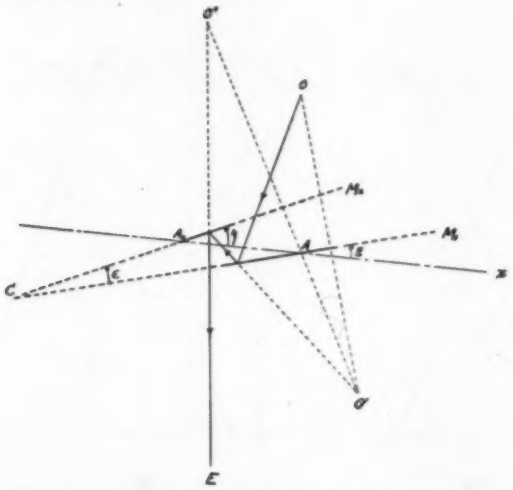

FIG. 3. *Two plane mirrors variable independently about parallel axes, A, A_0.*

Only within a limited range of dimensions in the apparatus can solutions be obtained graphically with any convenience, and at best these are approximate. For example, it is desirable to be able in such an instrument to place the object at any distance, including infinity. Exact analytical expressions were therefore sought.

LOCUS OF SECONDARY IMAGE

The locus of the secondary image for the case of two mirrors has been given.[2] In Fig. 3, reproduced from the former paper, M_1, M_2

[2] J.O.S.A. & R.S.I., *17*, 5, November, 1928, pp. 369-78. A general treatment of a system of any number of plane mirrors, in matrix mathematics, was published recently by T. Smith, Transactions of the Optical Society, *30*, 2, 1928–9, pp. 68–78.

are the two mirrors in principal section, with their independently variable angles, ξ, η, respectively, axes at A, A_0, and point-object O at a known position. M_1 represents an inner mirror, say for eye E_1, Fig. 1, and M_2 its outer mirror. With origin at A_0, put $O \equiv (x_0, y_0)$, $O' \equiv (x', y')$, $O'' \equiv (x'', y'')$, $\overline{A_0A} = d$. Then the secondary image O'' is located in polar coordinates by $\rho \equiv A_0O''$, $\phi \equiv \sqrt{AA_0O''}$, as follows (loc. cit., p. 374):

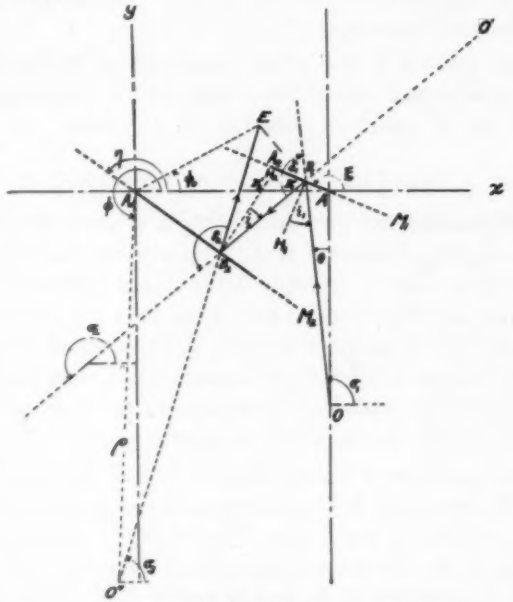

FIG. 4. *Derivational diagram.*

$$\left.\begin{aligned} \rho^2 &= (x_0^2+y_0^2)+2d[\,y_0 \sin 2\xi+2(d-x_0)\sin^2\xi\,] \\ \phi &= \eta+\alpha+\epsilon-\psi \end{aligned}\right\} \qquad (1)$$

where $\sin \alpha = [y_0 \cos \xi+(d-x_0)\sin \xi]/[(d-x_0)^2+y_0^2]^{1/2}$,

or $\alpha = \pi-\mu-\xi$,

where $\mu = \sqrt{OAA_0}$, $\epsilon = \eta-\xi$,

$$\sin \psi = d\,[y_0 \cos 2\xi+(d-x_0)\sin 2\xi]/[\rho[(d-x_0)^2+y_0^2]^{1/2}].$$

(Further, $x' = d+r \cos(\xi-\alpha)$, $y' = r \sin(\xi-\alpha)$, also $x'' = \rho \cos \phi$, $y'' = \rho \sin \phi$.)

PROBLEM OF THE FIELD OF VIEW FOR THE SPECIFIED EXPERIMENTAL CASE

In the sense that it regarded the reflecting planes as also transmitting, the derivation of the locus equations just given was general. Moreover, it did not deal with the angular limits of visibility (that is, "field of view" in the present sense) of the secondary image for stated dimensions in an apparatus (loc. cit. p. 375). The detailed solution is given now, for this instrument.

The present problem is that of the boundaries of the "field of view," that is, the angular field within which image O'' is everywhere visible to the eye at E, Fig. 4, conditions being (a), (b), (c), above.

DETAILED STATEMENT OF PROBLEM

In the oclinometer the first mirror M_1 is given a definite limited width, $\overline{AA_e} = w_1$, Fig. 4, where $A_e \equiv (x_e, y_e)$, and $x_e = w_1 \cos \xi$, $y_e = w_1 \sin \xi$, its angle ξ is kept constant at 160°, object O is placed below the x-axis of coordinates, directly beneath axis A, so that $\mu = \sqrt{OAA_0} = -90°$, fixed point $E \equiv (x_3, y_3)$ above that axis, and η remains variable so as to throw the emergent ray B_2E, corresponding to a chosen incident ray OB_1, through point E. Likewise, at the outset, the lower, or inner, edge of M_2 is taken at axis A_0, its width w_2 undetermined.

With these conditions it is seen, Figs. 5, 6, that the upper, or outer, edge A_e of M_1 determines both the lower and the upper limits (that is, limiting rays R_1, R_2) of the field of view of O'' to an eye at E. Let σ_1, σ_2, σ_3, Fig. 4, be the slope angles of the respective ray segments, incident OB_1, transmitted B_1B_2, and emergent B_2E. Then the magnitude of the field of view, Fig. 7, is $\Delta\sigma = (\sigma_3)_2 - (\sigma_3)_1$. The field of view is identical in location with the emergent visible pencil converging at E with η variable.

But it is not enough merely for $\Delta\sigma$ to retain a sufficiently high value for all values of object distance (from minimum distance of distinct vision to infinity inclusive). At the same time, it is necessary that the "direction of the field of view," that is, the direction of its bisector, shall remain in the neighborhood of that of the longitudinal axis of the system. This direction is $\sigma_v = [(\sigma_3)_1 + (\sigma_3)_2]/2$.

The lower limit of the field of view, ray R_1, having slope angle $(\sigma_3)_1$, Fig. 5, is known at the outset, since R_1 is that ray which, as incident ray, $O(B_1)_1$, is just about to graze M_1 at its outer edge A_e. Now $(\sigma_3)_1$ is

determined by $(\sigma_1)_1 \equiv \theta_1 + 90°$, where $\theta_1 \equiv \sphericalangle AOA_e$, and $(B_1)_1 \equiv A_e$, where θ_1 and A_e are given. *Then σ_3 must be found as an extrinsic function of θ in order to obtain $(\sigma_3)_1$ from θ_1 (see eqs. 2, 3 below).*

Also, the upper limit of the field of view, ray R_2, Fig. 6, is determined by itself just grazing edge A_e of M_1. This limiting ray has slope angle $(\sigma_3)_2 = (\alpha)_1$, the known slope angle of line A_eE.

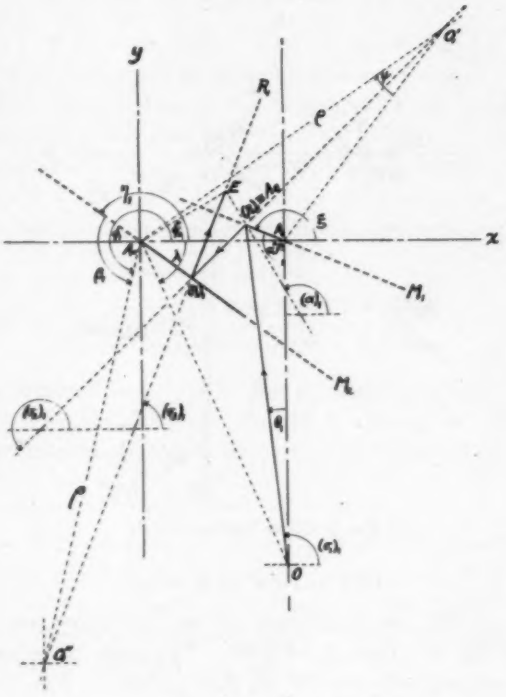

FIG. 5. *Lower limiting ray, R_1.*

Although the two angles $(\sigma_3)_1$, Fig. 5, $(\sigma_3)_2$, Fig. 6, when both are known, are sufficient for finding the magnitude of the field of view, $\Delta\sigma$, and also its bisector, σ_v, it is desirable to know further the useful areas of the two mirrors with the present conditions, in order to find the necessary positions (points B, below) of their edges. These areas are fixed by the two pairs of extreme incidence points: for the first mirror, A_e, $(B_1)_2$, Figs. 5, 6; and for the second mirror, $(B_2)_1$, $(B_2)_2$. Of these four points, A_e is given, $(B_1)_2$ depends upon the unknown θ_2, $(B_2)_1$

upon the given θ_1, but also upon the corresponding, unknown η_1, and $(B_2)_2$ upon the known $(\sigma_3)_2$ and the unknown θ_2. *In order to know* θ_2, θ *must be found as an extrinsic function of* σ_3 (*see eqs.* 2, 4, 5, *below*).

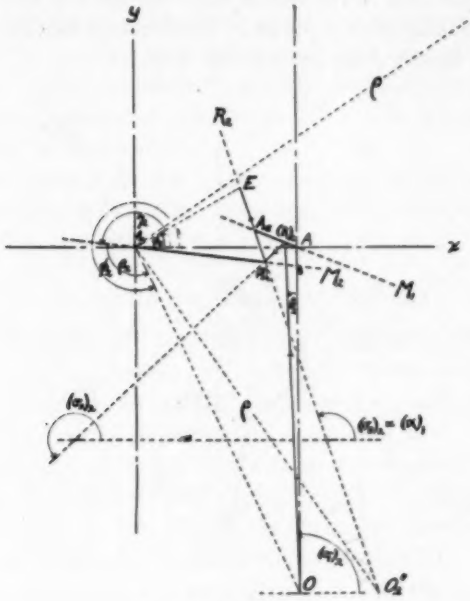

FIG. 6. *Upper limiting ray,* R_2.

DERIVATION OF $\sigma_3 = f(\theta)$.

To find $\sigma_3 = f(\theta)$ two equations, (2) and (3) below, are used. Put $\sphericalangle AOB_1 \equiv \theta$, Fig. 4. Then $\sigma_1 = \theta + 90°$. The angle of incidence on M_1 at $B_1 \equiv (x_1, y_1)$ is $i_1 = \theta - \xi + 180$, so that $\sphericalangle B_2B_1O = 2(\theta - \xi + 180)$, and the first deviation is $\delta_1 = -(180 - \sphericalangle B_2B_1O) = 2(\theta - \xi - 90)$, and $\sigma_2 = \sigma_1 - \delta_1 = (2\xi - \theta) - 90 = \tau - 90$, where $\tau = 2\xi - \theta$. The angle of incidence on M_2 at $B_2 \equiv (x_2, y_2)$ is, from $\Delta B_2B_1'B_2'$, $i_2 = -(\eta - 90) - (\sigma_2 - 180) = \tau - \eta - 180$, while the second deviation is $\delta_2 = 180 - 2(-i_2) = 2(\tau - \eta) - 180$. Then

$$\sigma_3 = \sigma_2 - \delta_2 = -\tau + 2\eta + 90. \qquad (2)$$

Either θ or σ_3 can be regarded as variable independently, η being considered an unknown dependent variable.

Since eq. (2) involves two unknowns, θ and η, a second eq. is needed.

From ΔEA_0B_2, Fig. 4, $\overline{B_2E}/\sin(180-\eta+\phi_0)=\overline{A_0E}/\sin(90+i_2)$, or $\overline{B_2E}/\sin(\eta-\phi_0)=\overline{A_0E}/-\cos(\tau-\eta)$. Also, from ΔEB_2B_1, $\overline{B_2E}/\sin(\sigma_2-(\alpha))=\overline{B_1E}/\sin(-2i_2)$, or $\overline{B_2E}/\cos(\tau-(\alpha))=\overline{B_1E}/\sin 2(\tau-\eta)$, $(\alpha)\equiv\sqrt{}(180^\circ-\text{slope angle of } B_1E)$. Eliminating $\overline{B_2E}$ and solving for η,

$$\sin\eta=\pm\left(\frac{-B\pm(B^2-4AC)^{1/2}}{2A}\right)^{1/2}, \tag{3}$$

where $A=-1$, $B=1-\cos(\phi_0+\tau)\,[\cos(\phi_0+\tau)+2(a+\sin\phi_0\sin\tau)]$, $C=-[a^2+(2a+\sin\phi_0\sin\tau)\sin\phi_0\sin\tau]$, where $a=-[\overline{B_1E}\cos(\tau-(\alpha))]/(2\overline{A_0E})$, $\overline{B_1E}=[(x_2-x_1)^2+(y_2-y_1)^2]^{\frac{1}{2}}$, and $\tau=2\xi-\theta$, $\tan\theta=(d-x_1)/(y_1-y_0)$, $\tan\phi_0=y_2/x_2$, $\tan(\alpha)=(y_2-y_1)/(x_2-x_1)$, $\overline{A_0E}=\sqrt{x_2^2+y_2^2}$, $x_1=d-r/(\tan\xi+\cot\theta)$, $y_1=-r/(1+\cot\xi\cot\theta)$, where $r=AO=-y_0$ here.

Then $\sigma_3=f(\theta)$ by elimination of η between the two equations

$$\sigma_3=\theta+2\eta-2\xi+90, \tag{2 bis}$$

$$\sin\eta=\pm\left(\frac{-B\pm(B^2+4C)^{1/2}}{-2}\right)^{1/2}, \tag{3 bis}$$

and thus is obtained $(\sigma_3)_1=f(\theta_1)$. But attempting to solve (2), (3), extrinsically for $\theta=f(\sigma_3)$, in order to get $\theta_2=f(\sigma_3)_2$, we come upon a difficult function, and so look for an expression to be derived differently.

DERIVATION OF $\theta=f(\sigma_3)$

Three equations are used to find $\theta=f(\sigma_3)$: eq. (2) above; also a relation containing η and ϕ (where ϕ is directional $\sqrt{}AA_0O''$ of the second image O''), derived in the preceding paper (loc. cit., p. 373): and third, an expression for sin ϕ dependent on σ_3. These three eqs., which can be combined to eliminate η and ϕ, are

$$\theta=\sigma_3-2\eta+2\xi-90, \tag{2 bis}$$

$$2\eta=\phi+\xi-\alpha+\psi, \tag{4}$$

$$\sin\phi=-(b\cos\sigma_3\pm\sin\sigma_3(1-b^2)^{1/2}), \tag{5}$$

where α is the same as in eqs. (1),

$$\sin\psi=d[y_0\cos 2\xi+(d-x_0)\sin 2\xi]/[\rho[(d-x_0)^2+y_0^2]^{1/2}],$$

$b=\overline{A_0E}\sin(\sigma_3-\phi_0)/\rho$, and ρ is the same as in eqs. (1). Eq. (5) is obtained from $\sqrt{}A_0O''E$, Fig. 4, in which $\overline{A_0E}/\sin(\phi-\sigma_3+180)=\rho/\sin(\sigma_3-\phi_0)$. (Here again if the reverse solution $\sigma_3=f(\theta)$ extrinsically is attempted from eqs. 2, 4, 5, another difficult function is met).

USEFUL AREAS OF THE MIRRORS

Knowing now for a given experimental case all four quantities, θ_1, $(\sigma_3)_1 = f(\theta)_1$, and $\theta_2 = f(\sigma_3)_2$, $(\sigma_3)_2$, and associated quantities, we can then know also all the coordinates of the corresponding points of incidence, $(B_1)_1$, $(B_2)_1$, and $(B_1)_2$, $(B_2)_2$, for the two limiting rays, and so know the useful areas of the mirrors.

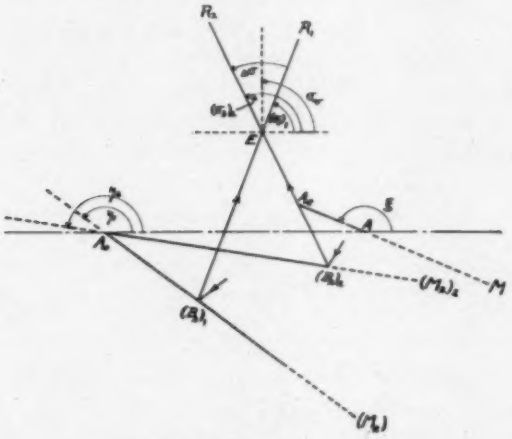

FIG. 7. *Field of view in oclinometer.*

The two incidence points, $B_1 \equiv (x_1, y_1)$, $B_2 \equiv (x_2, y_2)$, Fig. 4, are located respectively at the intersections of two pairs of straight lines: (1) AB_1 (first mirror), $y = (x-d) \tan \xi$; (2) OB_1 (incident ray), $y = y_0 - (x-d) \cot \theta$; and (3) B_1B_2 (transmitted ray), $y = y_1 - (x-x_1) \cot \tau$; (4) A_0B_2 (second mirror), $y_2 = x_2 \tan \eta$. From (1), (2), $x_1 = d + y_0/(\tan \xi + \cot \theta)$, $y_1 = y_0/(1 + \cot \xi \cot \theta)$. From (3), (4), after putting $y = y_2$, $x = x_2$ in (3), $x_2 = (y_1 + x_1 \cot \tau)/(\cot \tau + \tan \eta)$, $y_2 = x_2 \tan \eta$. Point B_2 depends upon η, corresponding to a given value of θ, or of σ_3.

The boundaries of the useful areas of the two mirrors are then as follows:

for M_1,— (a) outer edge, $(B_1)_1 \equiv A_o$, Fig. 5;
(b) inner edge, $(B_1)_2 \equiv [(x_1)_2, (y_1)_2]$, Fig. 6,

where $(x_1)_2 = d + y_0/(\tan \xi + \cot \theta_2)$, $(y_1)_2 = y_0/(1 + \cot \xi \cot \theta_2)$, θ_2 being found from eqs. (2, (4, (5;

for M_2,— (a) inner edge, $(B_2)_1 \equiv [(x_2)_1, (y_2)_1]$, Fig. 5,

where $(x_2)_1 = [(y_1)_1 + (x_1)_1 \cot \tau_1]/(\cot \tau_1 + \tan \eta_1)$, $(y_2)_1 = (x_2)_1 \tan \eta_1$,

where $\tau_1 = 2\xi - \theta_1$, η_1 being found from eq. (2;

(b) outer edge, $(B_2)_2 \equiv [(x)_2, (y_2)_2]$, Fig. 6,

where $(x_2)_2 = [(y_1)_2 + (x_1)_2 \cot \tau_2]/(\cot \tau_2 + \tan \eta_2)$, $(y_2)_2 = (x_2)_2 \tan \eta_2$, and $\tau_2 = 2\xi - \theta_2$, η_2 also being found fom eq. (2.

The widths of the useful areas of M_1, M_2, are then respectively $W_1 = w_1 - \overline{A(B_1)_2} = w_1 - \sqrt{[d-(x_1)_2]^2 + (y_1)_2^2}$ and $W_2 = w_2 - \overline{A_0(B_2)_1} = w_2 - \sqrt{(x_2)_1^2 + (y_2)_1^2}$, where $w_1 = \overline{AA_e}$, $w_2 = \overline{A_0(B_2)_2} = \sqrt{(x_2)_2^2 + (y_2)_2^2}$.

DEPENDENCE OF FIELD OF VIEW FOR DIFFERENT OBJECT DISTANCES, UPON THE DIMENSIONS CHOSEN IN THE INSTRUMENT

As mentioned already, a practicable instrument for the measurement and correction of muscular imbalance requires, *for all object distances*, at least three things,—large field of view, $\Delta\sigma$, approximate constancy, as well as desired value, of its direction, σ_v, and compactness. In Table 1

TABLE 1.

Values of field of view, $\Delta\sigma$, and its direction, σ_v, roughly approximate, obtained graphically for different values of at least six variable quantities.

Eye position								
x_2 (cm)	y_2 (cm)	ξ (°)	d (cm)	$-y_0$ (cm)	w_1 (cm)	$\Delta\sigma$ (°)	σ_v (°)	Remarks
1.8	1.3	45	5	20	5.6	31	100	Rays incid. first on outside mirror
6.8	1.3	45	10	20	5.6	47	107	Rays incid. first on outside mirror
12.0	1.3	45	15	20	5.6	52	109	Rays incid. first on outside mirror
5.2	4.0	135	10	20	5.6	31	30	Rays incid. first on M_1
2.5	4.0	135	6	20	4.2	44	37	
2.5	2.4	160	6	20	3.4	53	86	
2.5	2.4	160	6	30	3.4	59	84	
2.5	2.4	160	6	40	3.4	57	85	
2.8	3.0	160	6	20	2.8	36	90	
2.8	3.0	160	6	20	2.3	48	101	
2.8	3.0	160	6	20	2.3	41	87	Axis A_0 raised 1.2 cm above x-axis
2.8	3.0	180	6	20	1.8	21	104	
4.8	3.0	160	8	20	2.3	43	85	Axis A_0 raised 0.6 cm above x-axis
2.8	3.0	160	6	20	2.3	38	95	Axis A_0 raised 0.6 cm above x-axis
2.8	3.0	160	6	20	2.3	36	98	Axis A_0 dropped 0.6 cm below x-axis
5.8	3.0	160	9	20	2.3	48	90	
2.8	3.0	150	6	20	2.3	36	73	Too great dispersion at M_1
2.8	3.0	170	6	20	2.3	13	103	Outer part only of M_1 utilized
5.8	3.0	170	9	20	2.3	21	98	
5.8	3.0	160	9	25	2.3	49	91	
5.8	3.0	160	9	30	2.3	48	92	Practicable
5.8	3.0	160	9	∞	2.3	41	94	

are given roughly approximate graphical results for definite values of at least six variable quantities, including object distance y_0, in the optical system. These were obtained in attempts to reach a satisfactory design. *The last three of the twenty-two different cases meet the requirements.* In these cases the only variable is y_0, while $\Delta\sigma > 40°$, and σ_v does not depart too far from 90°. Dimensions in the instrument were therefore chosen accordingly.

APPLICATION OF THE DERIVED EQUATIONS TO THE PARTICULAR EXPERIMENTAL CASE

The particular dimensions, based on the last three cases of Table 1, chosen for the present instrument, are included in the following numerical values of geometrical quantities that appear in the eqs.: (a) angles,—$\xi = 160°$, $\mu = -90$ (Fig. 5), $\alpha = 110$, $\phi_0 = 27°\ 9'$ ($\tan\phi_0 = 0.51282$), $(\phi_0 + \tau) = 347°\ 9' - \theta$; (b) linear distances (all in cm),—$d = 9$, $x_3 = 5.8500$, $y_3 = 3$, $x_e = 6.8388$, $y_e = 0.7866$ (since $w_1 = 2.3$ cm), $w_2 = 8$ approximately, $x_0 = 9$, $y_0 = -r$ (a variable parameter[3]). Since only negative values of y_0 are considered (point O always below x-axis), $(\sigma_3)_2 = (\alpha)_1 = \text{const.}$, where $\tan(\alpha)_1 = (y_3 - y_e)/(x_3 - x_e) = -2.23865$, so that $(\sigma_3)_2 = 114°\ 4'18''$. With these dimensions, numerical values of y_0 can vary over a range corresponding to lengths of total ray-path $(\overline{OB_1} + \overline{B_1B_2} + \overline{B_2E})$ extending from minimum distance of distinct vision (taken as 25 cm)[4] to infinity, while $\Delta\sigma$ and σ_v simultaneously retain desirable values that are approximately constant.

Then from eqs. (2), (3), (4), (5) are obtained the *working formulas*: (a) for $\sigma_3 = f(\theta)$,

$$\sigma_3 = \theta + 2\eta - 230°, \tag{6}$$

$$\sin\eta = \pm\left(\frac{-B \pm (B^2 + 4C)^{1/2}}{-2}\right)^{1/2}, \tag{7}$$

where

$$B = 1 - \cos(347°\ 9' - \theta)[\cos(347°\ 9' - \theta) + 2[a + 0.45632\sin(320° - \theta)]],$$

$$C = -[a^2 + [2a + 0.45632\sin(320° - \theta)]\cdot 0.45632\sin(320° - \theta)],$$

$$a = -[\overline{B_1E}\cos(320° - \theta - (\alpha))]/13.148,$$

$$\overline{B_1E} = [(5.8500 - x_1)^2 + (3.0000 - y_1)^2]^{1/2},$$

$$\tan\theta = (9 - x_1)/(y_1 - y_0), \quad \tan(\alpha) = (3 - y_1)/(5.8500 - x_1),$$

$$x_1 = 9 + y_0/(\cot\theta - 0.36397), \quad y_1 = y_0/(1 - 2.74748\cot\theta);$$

[3] Quantities $r = \overline{AO}$, $d = \overline{A_0A}$ are considered intrinsically positive.

[4] While 25 cm may be an objectionable average value, it serves as a suitable lower limit here. If a larger value is taken, the conclusion as to relative constancy of $\Delta\sigma$, σ_v, still holds.

(b) and for $\theta = f(\sigma_3)$,

$$\theta = \sigma_3 - 2\eta + 230^\circ, \tag{8}$$

$$2\eta = \phi + \psi + 50, \tag{9}$$

$$\sin\phi = -[b\cos\sigma_3 \pm \sin\sigma_3(1-b^2)^{1/2}], \tag{10}$$

where $\sin\psi = -6.89436/\rho$, $b = 6.574\sin(\sigma_3 - 27^\circ\ 9')/\rho$, $\rho^2 = (81+y_0^2) - 11.57022\ y_0$.

Also, $O' \equiv (x',\ y')$, where $x' = d - y_0\cos(\xi-\alpha) = 9 - 0.64279\ y_0$, and $y' = -y_0\sin(\xi-\alpha) = -0.76604\ y_0$.

Then $(\sigma_3)_1$ is obtained from (6), (7), by using $\tan\theta_1 = 2.1612/(0.7866 - y_0)$, since here $x_1 = x_e$, $y_1 = y_e$. Also, θ_2 is found from (8), (9), (10), by substituting $(\sigma_3)_2 = 114^\circ\ 4'18''$. Angle $(\sigma_3)_1$ depends upon y_0, but $(\sigma_3)_2$ remains constant.

TABLE 2.

$-y_0$ (cm)	θ_1	$(\sigma_3)_1$	$(\sigma_3)_2$	$\Delta\sigma$	σ_v
20	5° 56′ 8″	66° 39′ 52″	114° 4′ 18″	47° 24′ 26″	90° 22′ 5″
50	2 26 10	70 32 6	114 4 18	43 32 12	92 18 12
∞	0 0 0	73 12 0	114 4 18	40 52 18	93 38 9

APPROXIMATE CONSTANCY OF FIELD OF VIEW FOR PRESENT DIMENSIONS IN THE OCLINOMETER, OVER THE COMPLETE RANGE OF OBJECT DISTANCES

In Table 2 are given precise solutions analytically for dimensions in the instrument indicated by the last three cases of Table 1. The range of object distances is, for the eye, 25 cm (approximately) < [total ray-path (0 to E)] < ∞, corresponding to 20 cm < $-y_0$ < ∞. *Because of this particular choice of combined dimensions, the field of view and its direction show comparatively low variability over this large range.*

CONCLUSION

An optical instrument using plane mirrors rather than refracting prisms, for the measurement and correction of muscular imbalance, for measuring the range of fusion in normal eyes, and in general permitting convergence independent of accomodation, was designed by J. R. Liggett and the writer. Requirements fulfilled by the design include: (1) all object distances from that corresponding to minimum distance of distinct vision, to infinity; (2) maximum projected width of inside mirrors less than interocular distance; (3) compactness; and,

for all object distances, (4) large field of view of secondary image, (5) approximate constancy of its direction, and (6) satisfactory agreement of this direction with that of the axis of symmetry in the instrument.

Incidental to the design, the writer has obtained geometrical expressions for (1) magnitude of field of view, or visible angular range, with point-object stationary, of the secondary image, (2) direction of the bisector of this field, and (3) boundaries of the useful areas of the mirrors. These expressions are applicable to a system of two independently variable plane mirrors having separate parallel axes located in their respective planes, in such an instrument as the oclinometer, described schematically in this paper, and corresponding optically to the short-base range finder with light rays reversed.

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THE ANALYSIS OF THE COLORS OBSERVED IN PHOTOELASTIC EXPERIMENTS

By R. V. Baud and W. D. Wright

In this paper an account of the production of the color fringes in photoelastic experiments is given in detail.

In Part I the equation governing the reduction in energy transmission for given wave lengths and retardations is developed.

In Part II some influences are discussed which modify the final light distribution but which, either owing to the present lack of experimental data or for some other reason, have been neglected in the calculations.

In Part III a detailed color analysis is made of the fringes produced in models with stresses giving retardations up to 2μ.

INTRODUCTION

In the past it has been customary to give little more than a qualitative description of the colors observed in photoelastic experiments, and it appeared desirable to obtain a more accurate quantitative expression to represent the observed phenomena. This paper describes the method of calculation that was adopted and presents the results that have been obtained.

The formation of the color fringes is briefly as follows: Assume a plane polarized beam of light incident perpendicularly on a transparent model which is subjected to external forces similar to those acting when the machine part or structure is in service. An image of the test specimen is formed on a screen by means of a projection lens. Between lens and screen is an analyzer. The strains introduced make the piece doubly refracting and the incident vibration at any point is partially or completely resolved into two perpendicular vibrations. One of the two rays travels faster through the model than the other; consequently, on emerging, there is a certain phase difference between the two vibrations and the light is therefore elliptically polarized. The amount of light that is transmitted by the analyzer will depend on the orientation of the ellipse and this in turn depends on the stress direction, the retardation produced in the specimen, and on the wave length of the light. Thus, for a given retardation, light of certain frequencies may be eliminated altogether while the light of other frequencies will be reduced in intensity. A given energy transmission, and hence a given color, will then correspond to a certain retardation and this is dependent on the strains or stresses, respectively. We thus see that the colors in the image give an indication of the stresses in the model. The color of a fringe is fre-

quently qualitatively described as being the complementary of the radiation that is completely cut out, but this is only a very approximate description as no cognizance is taken of the energy distribution of the remainder of the light. Further, for small retardations no visible radiations are completely eliminated and as the retardation increases the stage is soon reached when two or more wave lengths are cut out and in both these cases the simple explanation of the color as complementary to the particular wave length cut out, is no longer applicable. The only way to determine the color accurately is to calculate, for any given retardation, what the resulting energy distribution is and thence determine the color from the mixture curves of the spectrum.

To make the first part of the calculation, namely, to determine the energy distribution, it is necessary to know the law governing the reduction in energy transmission for given wave lengths and retardations. This law is derived in the first part of this paper; the second part contains a discussion of some complications encountered in the complete investigation; in the third part a color analysis is made on the basis of certain simplifying assumptions.

PART I

We will assume that the optical set consists of a monochromator, two crossed Nicols, two crossed quarter wave plates and a system of lenses. The vibration of a ray of plane polarized light, as transmitted by the first Nicol may be expressed by a light vector as given in equation (1):

$$\eta_0 = a_0 \cos \omega t \tag{1}$$

see Fig. 1. This vector is equivalent to two other vectors x, y combined, equations (2):

$$\left.\begin{aligned} x &= a \cos \omega t = a \sin (\omega t + \pi/2) \\ y &= a \cos \omega t \end{aligned}\right\} \tag{2}$$

where

$$a = a_0/\sqrt{2}.$$

We will assume that the axes of the quarter wave plates coincide with the x, y directions and furthermore that after the light has passed the plate, the ray in the x direction lags behind that in the y direction one quarter of the wave length; consequently the two vectors x, y will be as given by equations (3):

$$\left.\begin{aligned} x &= a \sin \omega t \\ y &= a \cos \omega t \end{aligned}\right\}. \tag{3}$$

Equations (3) represent circularly polarized light, the light vector rotating in clockwise direction as indicated in Fig. 1. This light is now incident upon the model. For simplification we will assume that the principal stress directions of the element considered coincide with the axes x, y of the mica plates. The retardation R in units of lengths produced in the model is given by Wertheim's law:

$$R = c(p - q)t \tag{4}$$

where p and q are the principal stresses of the element, t is the thickness

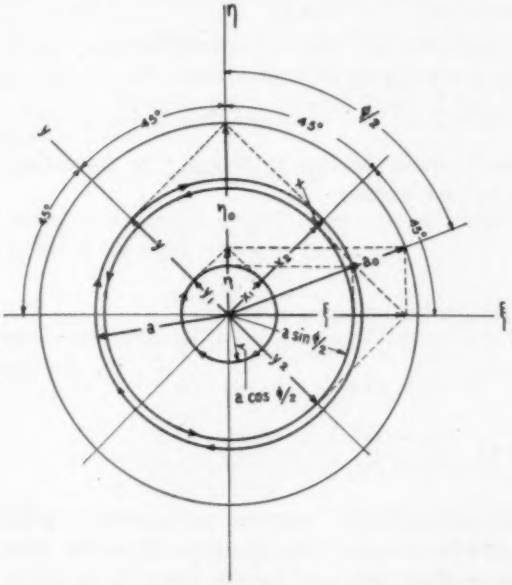

FIG. 1. *Schematical diagram of change in polarization of the light passing through a photoelastic instrument.*

of the specimen and c is a constant given as the difference $(n_1 - n_2)$ between the indices of refraction n_1 and n_2 of the refracted rays per unit stress.

The retardation may also be given in radians instead of units of lengths. In this case it is denoted by ϕ and given by equation (5):

$$\phi = \frac{R}{\lambda} 2\pi \tag{5}$$

where λ is the wave length of the light in the same units as the retardation R.

It is assumed further that the light represented by vector y of equation (3) is retarded with respect to that represented by vextor x, so that the light after passing through the model is as given by equation (6):

$$\left.\begin{aligned} x &= a \sin \omega t \\ y &= a \cos (\omega t - \phi) \end{aligned}\right\}. \tag{6}$$

These equations represent elliptically polarized light. They may also be written in the form:

$$\left.\begin{aligned} x &= a \sin [(\omega t - \phi/2) + \phi/2] \\ &= a \cos \phi/2 \sin (\omega t - \phi/2) + a \sin \phi/2 \cos (\omega t - \phi/2) \\ y &= a \cos [(\omega t - \phi/2) - \phi/2] \\ &= a \cos \phi/2 \cos (\omega t - \phi/2) + a \sin \phi/2 \sin (\omega t - \phi/2) \end{aligned}\right\}. \tag{7}$$

Equations (7) show the two vectors x, y as consisting each of the sum of two vectors, namely:

$$\left.\begin{aligned} x &= x_1 + x_2 \\ y &= y_1 + y_2 \end{aligned}\right\} \tag{8}$$

which vectors can be combined as follows:

$$\left.\begin{aligned} x_1 &= a \cos \phi/2 \sin (\omega t - \phi/2) \\ y_1 &= a \cos \phi/2 \cos (\omega t - \phi/2) \end{aligned}\right\} \tag{9}$$

and

$$\left.\begin{aligned} x_2 &= a \sin \phi/2 \cos (\omega t - \phi/2) \\ y_2 &= a \sin \phi/2 \sin (\omega t - \phi/2) \end{aligned}\right\}. \tag{10}$$

Equations (9) and (10) each represent a circularly polarized ray. The rotation of the ray as given by equations (9) is the same as that of the incident circularly polarized rays as given by equations (3). The rotation of the ray as given by equations (10) is in the opposite direction, see Fig. 1.

The circularly polarized rays as given by equations (9) and (10) are then incident upon the second quarter wave plate which affects the vectors y_1 and y_2 as indicated by equations (11) and (12):

$$\left.\begin{aligned} x_1 &= a \cos \phi/2 \sin (\omega t - \phi/2) \\ y_1 &= a \cos \phi/2 \cos (\omega t - \phi/2 - \pi/2) = a \cos \phi/2 \sin (\omega t - \phi/2) \end{aligned}\right\} \tag{11}$$

and

$$\left.\begin{aligned} x_2 &= a \sin \phi/2 \cos (\omega t - \phi/2) \\ y_2 &= a \sin \phi/2 \sin (\omega t - \phi/2 - [\pi/2]) = - a \sin \phi/2 \cos (\omega t - \phi/2) \end{aligned}\right\}. \tag{12}$$

It will be seen from equations (11) that the vectors x_1 and y_1 are now in time phase again and result in a vertically plane polarized ray as represented by vector η in Fig. 1 and of the magnitude as given by equation (13):

$$\eta = a_0 \cos \phi/2 \sin (\omega t - \phi/2). \tag{13}$$

Since the second Nicol is in crossed position with respect to the first so as to transmit only light polarized in a horizontal plane, the vertically plane polarized light as represented by equation (13) is completely obstructed.

The vectors x_2 and y_2 given by equations (12) are also in time phase, but in this case vector y_2 is negative and both vectors result, when added vectorially, in a horizontally plane polarized ray of the magnitude as given by equation (14):

$$\xi = a_0 \sin \phi/2 \cos (\omega t - \phi/2). \tag{14}$$

This light passes through the second Nicol without obstruction. The maximum value of the light vector is therefore:

$$\xi_{max} = a_0 \sin \phi/2. \tag{15}$$

As the energy is proportional to the square of this vector, the following equation results:

$$E = E_0 \sin^2 \phi/2 \tag{16}$$

in which E_0 is the energy incident on the specimen and E is the final energy transmitted through the analyzer.

In the above derivation of equation (16) it was assumed that the principal stress directions of the element coincide with the axes of the mica plates. It can be shown that equation (16) holds generally regardless of the principal stress directions, provided the mica plates are true quarter wave plates. This, however, is the case only for one specific wave length. There enter a few additional factors we have also to neglect in the present study, as will now be discussed in part II.

PART II

To simplify the problem on hand somewhat, it is necessary to neglect certain factors that in practice have a modifying influence on the resultant color fringes. Equation (16) has been derived for monochromatic light of such wave length that the mica plates are true quarter wave plates. More often, however, incandescent lamps emit-

ting a continuous spectrum are employed, instead of monochromatic light. There then enters a factor f into equation (16) which takes into consideration the fact that the light when it enters into the model is already elliptically polarized. This is necessary since the mica plates produce a quarter-wave retardation for only one wave length. For a balanced design of the optical instrument we may expect that the factor f may drop from unity to .8 at the most. In part III this factor is taken equal to unity, which is the correct value only either if we have light of the proper wave length or for the case that the principal stress directions of the element coincide with the axes of the mica plates, or both.

It is further assumed that c in equation (4) is strictly constant, but in reality being a function of the indices of refraction, it varies with the wave length. This relation will not affect the final color very materially but for a complete discussion it should be included. However, no data seem to be available to make this correction and it has therefore also been neglected.

A further approximation which affects the distribution insofar as the intensity is concerned but which has no effect on the color itself, concerns the amount of light that is affected by the double refraction. Equation (16) is based upon the assumption that the double refraction is complete regardless of the magnitude of the strains. It seems that for small strains this is not the case. A knowledge of the proportion of doubly refracted light to the complete light beam with increasing strain evidently is of considerable importance for a final color-analysis at least for the first fringes. Information on this point could be obtained by making observations on the color and intensity differences between the patterns obtained with crossed and parallel polarizing prisms. The problem should lend itself to both a theoretical and practical method of solution. A cursory examination was made by us which indicated that the effect of incomplete double refraction is inappreciable after the first fringe or two at the most. For the final color-analysis to be complete, quantitative results should be available.

There is yet another factor that has been neglected in our analysis; namely, the ordinary absorption in the material. This is sometimes considerable, especially at shorter wave lengths, as celluloid and bakelite are both far from being colorless. To make any correction for this would not be of much value as the absorption is a distinctly variable quantity, so that only the case for a wholly transparent substance has been worked out. Some disturbing effect of stray light upon the final color is also neglected.

The color-analysis was made for a light source that has an energy distribution similar to a black body radiating at 3000°K, see Fig. 2. This is a very close approximation to the radiation as produced by the tungsten-incandescent lamp used in the optical set in our Research Laboratories and the error introduced is much less than that involved, for instance, by neglecting the absorption in the model.

In view of the facts outlined above, it would not be surprising if the value of making any calculation at all were questioned. But these approximations are all of secondary importance and in themselves play no part in forming the color fringes. They merely exercise a modifying influence on the color quality. The main value of the following

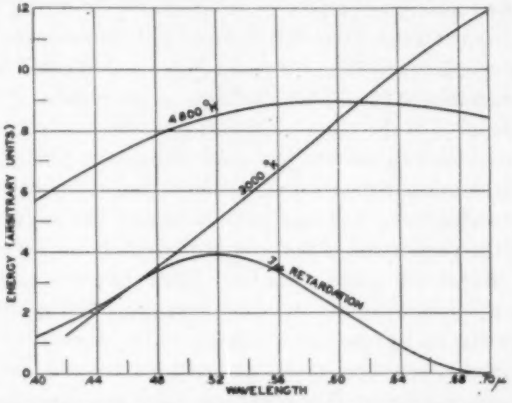

FIG. 2. *Energy distributions in the visible spectrum for black body radiations of 4800°K and 3000°K and for the latter radiation after transmission through the photoelastic apparatus in which is a model producing .7μ retardation.*

calculations is in giving a detailed account of the formation of the fringes and, of less importance, in providing an exact analysis of the colors for certain postulated conditions.

PART III

The final step in this investigation consists in converting the energy radiations into expressions to indicate the color sensations aroused. Whilst physically the energy distribution is the most fundamental method of defining the nature of a beam of light, it is not a satisfactory method of expressing the color quality of the light. Although color calculations are familiar to many and have been explained in various publications, it seems desirable to describe briefly in this connection the steps necessary to specify a color from its energy radiation.

We have used the trichromatic rather than the dominant hue and saturation method of color specification. To obtain the color on the trichromatic basis the result can be derived directly from the "equal area" mixture curves of the white light spectrum. These mixture curves express the amounts of three primaries that are required to match the spectral colors in the proportions in which they are present in the white light spectrum. The units of the primaries are based on the assumption that equal quantities are required to match white; thus, the mixture curves for white light must enclose equal areas. These curves are shown in Figure 3a; the areas between the curves and the horizontal axis are equal, portions below the axis of course being counted as negative. Any number of mixture curves can be drawn by using different references primaries; they will all represent the same set of facts but in different coordinates. The curves in Fig. 3 are referred to the monochromatic radiations .65μ, .53μ, and .46μ, as primaries. For any wave length the heights of the three ordinates give the amounts of the three primaries required to match the color sensation produced by that wave length, account being taken of the actual energy present in the white light radiation. For any new radiation the ordinates will be modified in the proportion of the energy present at each wave length in the new to that in the white radiation. Then the areas under the three curves give the proportions of the three primaries that would match the sensation produced by the new radiation. To express the result in a unit trichromatic equation, these proportions are reduced equally so that their sum is unity. This gives the color specification that is required. In addition the intensity of the final color fringe can be obtained by summing the products of the areas of each primary curve by the appropriate luminosities of unit quantities of the primaries.

In making a calculation the first requirement is a set of mixture curves for a known white light radiation. At present there are no standard data, so the values we have used are the result of some recent computations on the basis of a new set of trichromatic coefficients one of us determined experimentally a year or so ago in England.* The mixture curves (Fig. 3a) are given in a paper which has been communicated to the Optical Society (London) and will shortly be published by them under the title of "A Re-determination of the mixture curves of the spectrum" by W. D. Wright. The data are computed for a white light corresponding to a source with a black body radiation of 4800°K,

* W. D. Wright, "*A Re-determination of the Trichromatic Mixture Data*," Medical Research Council, Special Report Series, No. 139.

which is the standard used at present by the National Physical Laboratory. The primaries, as mentioned above, are the spectral radiations .65μ, .53μ, and .46μ. These have only been used as temporary reference primaries until standard primaries have been agreed upon by the various national laboratories. That, however, does not diminish their value for our present purpose.

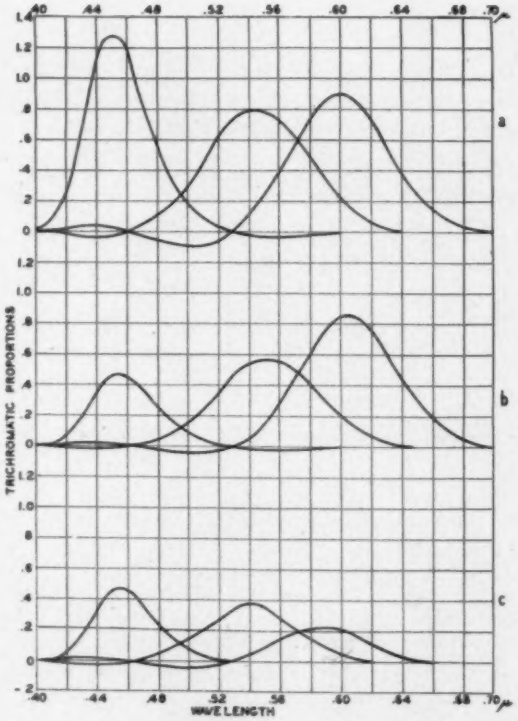

FIG. 3. *Three sets of color mixture curves of the spectrum (spectral primaries. 65μ, .53μ and .46μ) given by the three energy distributions shown in Figure 2.*

(a) "Equal Area" Curves for black body radiation of 4800°K (standard white) curves.

(b) Curves for black body radiation of 3000°K.

(c) Curves for Black Body radiation of 3000°K after transmission through the photoelastic apparatus in which is a model producing .7μ retardation.

It is then necessary to compute the mixture curves for our assumed radiation, namely 3000°K, with the result shown in Fig. 3b. These were obtained from those in Fig. 3a by reducing the ordinates at suitable wave length intervals in the ratio of the energy of the 3000°K radiation

to that of the 4800°K radiation for each wave length. These energy radiations are shown in Fig. 2. The areas under the mixture curves in Fig. 3b are in the ratio of:

red 78.0 green 58.5 blue 30.0

Reducing these so that their sum is unity, the color equation for 3000°K becomes:

$$C = .469R + .351G + .180B.$$

The next problem is to determine the energy distribution after this incident beam has passed through the specimen and analyzer, consisting of prism and mica plate, when a strain has been given to the model to produce some given retardation. As an example consider the case when a retardation of .7μ has been introduced and calculate the effect on light of a wave length of .5μ. Then from equation (5):

$$\frac{\phi}{2} = \frac{R}{\lambda}\pi = \frac{.7}{.5}\pi$$

whence

$$\sin^2\frac{\phi}{2} = .904.$$

At .5μ the modified mixture data for 3000°K have the values:

red $-.045$ green .150 blue .090.

These values must be reduced by the factor .904, so that the new mixture values for .5μ become:

red $-.041$ green .136 blue .081

Similar calculations are repeated for other wave lengths at suitable intervals, all for the same retardation of .7μ until sufficient values are available to plot the mixture curves shown in Fig. 3c and, incidentally, the energy curve shown in Fig. 2 to give the mixture data for the retardation. The areas under these curves then have to be found and in this case are:

red 1.75 green 3.25 blue 2.8

giving a unit color equation of:

$$C = .224R + .417G + .359B.$$

The relative luminosity of the beam at this point is most simply found, with the values already obtained, by multiplying the areas under the

three curves by the respective relative luminosities of unit quantities of the three primaries and summing the products. These unit luminosities are:

$$L_r = 38.5 \qquad L_g = 100.0 \qquad L_b = 3.72,$$

so that the luminosities for $.7\mu$ retardation are proportional to:

red 67 green 325 blue 10

giving a total luminosity of 403. That completes the calculations for that retardation; the same process then has to be repeated for other retardations until sufficient information is obtained.

For our purpose calculations were made for retardations up to 2μ at intervals of $.1\mu$ with an extra value calculated at $.55\mu$ owing to the rapid transition of the color at that point. The results are given in the

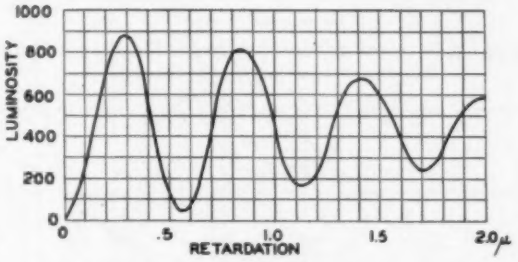

FIG. 4. *Total luminosity of the transmitted light for increasing retardations.*

accompanying table. For each retardation the areas under the mixture curves and the trichromatic coefficients, which specify the color for that retardation, are given. In the last column the luminosity of the fringes corresponding to the various retardations are given.

The most instructive picture of what is taking place is given by the curves in Figures 4 and 5. In Figure 4 the luminosity of the fringes is plotted against the retardation. This curve shows clearly the periodic rise and fall in the intensity and also the gradual decrease in the amplitude of the intensity variation. Eventually the luminosity will reach a steady level. Figure 5 shows that much the same process is taking place with the individual primaries and that they too will reach a steady level. The values plotted in this diagram are the areas of the mixture curves. The proportions of the primaries finally become practically constant and the image of the specimen becomes of uniform hue and brightness. It is interesting to note that this stage is reached when the

retardation is so great that for a large number of wave lengths the light is completely eliminated. This is clearly shown in Figure 6, where the energy curves are shown both for the incident radiation and for the case of retardation of 20μ.

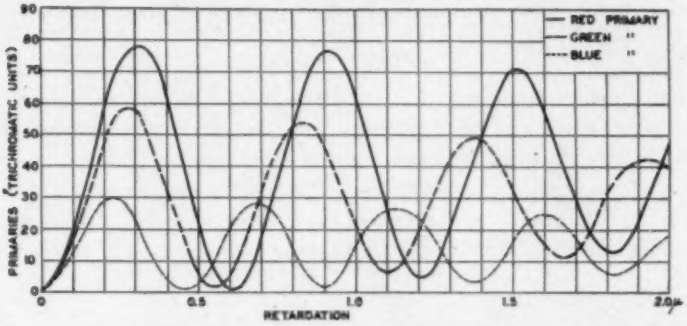

FIG. 5. *Amounts of the spectral primaries (wave lengths .65μ, .53μ and .46μ) required to match the colors corresponding to given retardations.*

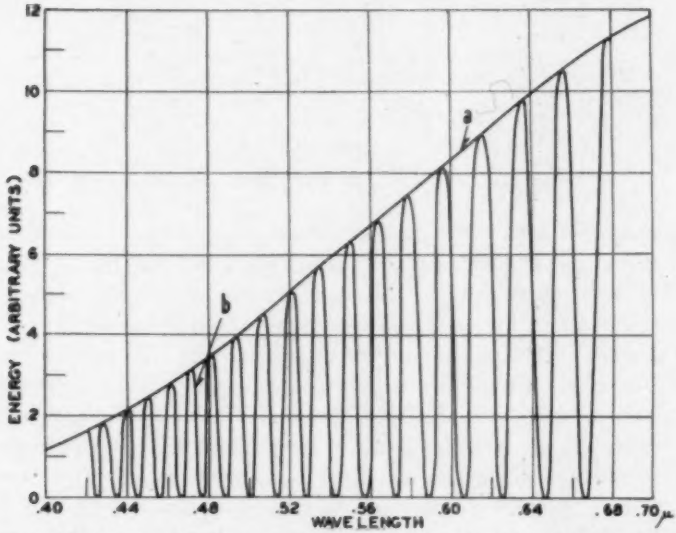

FIG. 6. (*a*) *Energy distribution in the visible spectrum for a black body radiation of 3000°K.* (*b*) *Same radiation as* (*a*) *but after transmission through the photoelastic apparatus in which is a model producing 20μ retardation.*

It is easy to see from Figure 5 how the proportions of the primaries vary rapidly from retardation to retardation, so that the colors show

similarly rapid changes. The final stage in the work is the representation of the color changes diagrammatically. This can be done by plotting the color coefficients in the color triangle, with the result shown in Fig. 7.

Before discussing Fig. 7 it may be useful to recount briefly the main properties of the color triangle. Although a unit color equation expresses any color in terms of three primaries, there are really only two independent variables since the sum of the coefficients is unity. Hence colors can be plotted in a two dimensional diagram using rectangular coordinates, namely, the red and green coefficients. Any color

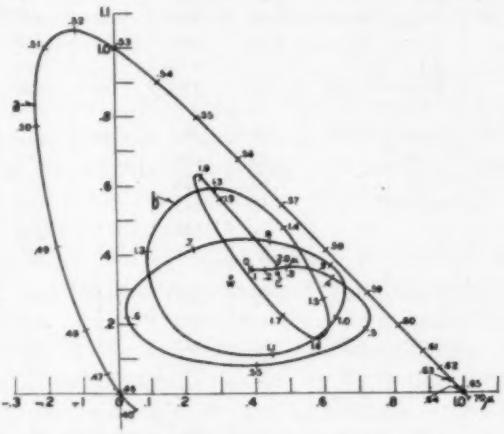

FIG. 7. *Color triangle using spectral primaries .65μ, .53μ and .46μ.*
(a) Locus of the spectral colors.
(b) Locus of colors corresponding to given retardations.
(W) White point, corresponding to the 4800°K radiation.
(L) Color of the incident light, namely, the 3000°K radiation.

is uniquely represented by a point in the diagram and the result of mixing two colors is found in a similar manner to the determination of the center of gravity of two separated bodies when the number of trichromatic units is substituted for the masses of the bodies. To get a good mental picture of the quality of the color represented by any given point, its relation to the white point *W* and the spectral colors must be considered. The spectral locus is given by the outer curve in Fig. 7. Any point lying between white and, say, .60μ will be a more or less pale orange according to whether it lies nearer to the white

point or to the spectral color. A closer correspondence between a point in the color field and a particular color sensation can only be realized after considerable familiarity with the practice of three color mixing.

Results of color analysis for increasing retardations

Retardation (μ)	Areas under the mixture curves			Trichromatic coefficients			Luminosity
	Red (.65μ)	Green (.53μ)	Blue (.46μ)	Red	Green	Blue	
0.0	—	—	—	.384	.356	.260	—
.1	1.9	1.7	1.2	.396	.354	.250	248
.2	5.8	4.8	2.9	.430	.355	.215	714
.3	7.8	5.8	2.3	.490	.365	.145	889
.4	6.1	3.3	.4	.622	.337	.041	566
.5	2.15	.56	.28	.719	.187	.094	140
.55	.70	.14	.90	.403	.080	.517	44
.6	.80	.60	2.05	.029	.220	.751	71
.7	1.75	3.25	2.80	.224	.417	.359	403
.8	5.4	5.3	1.45	.445	.436	.119	792
.9	7.7	4.6	.14	.618	.371	.011	761
1.0	6.15	2.1	1.5	.631	.215	.154	453
1.1	2.6	.65	2.55	.448	.112	.440	174
1.2	.45	2.0	2.4	.093	.412	.495	226
1.3	2.05	4.25	.85	.287	.594	.119	507
1.4	5.0	4.9	.4	.485	.475	.040	684
1.5	7.15	3.25	1.6	.596	.271	.133	606
1.6	5.7	1.55	2.5	.585	.159	.256	383
1.7	2.8	1.3	1.75	.478	.222	.300	244
1.8	1.25	3.15	.64	.248	.625	.127	365
1.9	2.25	4.2	1.0	.303	.563	.134	532
2.0	4.8	3.9	1.85	.456	.369	.175	582
3000°K radiation	7.8	5.85	3.0	.469	.351	.180	896

The complex curve in the triangle gives the locus of the colors corresponding to increasing retardations. By following the curve round, the normal color changes can be traced out. Thus we can check the observed color fringes which start at a neutral grey, turning to yellow, then orange to purple and a dark desaturated blue, then from a blue green to a pale green, thence to yellow and so the cycle of color changes continues but does not repeat. The loops in the curve will become smaller and smaller as the retardation increases, until they practically

coincide with the point representing the color of the incident light, point L in Fig. 7. Of course, there is no connection between the spacing of the points in Fig. 7 and the width of the color fringes in the image of the model. The width of the fringes depends on the distribution of the stress in the specimen; in some cases this may vary uniformly, in which case we have a set of parallel, equally spaced fringes; in other cases the pattern of the fringes may be very much more complicated. In neither case has Fig. 7 any connection with the pattern. It shows only the color quality of the fringes. One purpose of this diagram is that with it, it is possible to say what the color will be for a given retardation. More important, it will determine the retardation when the color is known. Without some accurate color measuring device it might be difficult to determine the color very closely, but by combining the results in Fig. 4 with Fig. 7 to get the approximate retardation from the number of dark or light fringes that exist between the point being considered and the neutral fringe, it is possible to get a very reliable value for the retardation from qualitative examination of the color. Thus in the third dark fringe we could say that the part with a purple color had a retardation of 1.1μ.

In conclusion it does not seem likely that the main value of this paper lies in its immediate practical application but rather in providing a more detailed description of the underlying processes involved in photoelasticity. Before anything of great practical use can be forthcoming, further information is required on the optical properties of the material used in the models. It may be that the results of this paper will help in obtaining this information. Finally, this paper illustrates the feasibility of quantitative, as opposed to qualitative color specifications, an advantageous process which could be made more use of in other branches of physics and optics.

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ULTRAVIOLET RADIATION FROM THE SUN AND HEATED TUNGSTEN

By W. E. Forsythe and Frances Christison

At the present time there is considerable interest in the production of ultraviolet radiation in the particular wave-length region that has been found to be of some therapeutic value. There is not much information available concerning just how much ultraviolet radiation or just what wave-length limits are necessary to produce the desired therapeutic effects. Maughan in a recent paper[1] reviewed this subject and gave some very interesting data on the wave-length limits of the radiation necessary to cure rickets in chickens. This author showed that the mercury line 2967A is much more potent for this effect than 3020A which is more intense. According to his work energy of wave lengths longer than 3020A have very little, if any, value in curing rickets in chickens. Maughan showed that irradiating the chickens with a mercury arc through window glass for about 22 minutes a day had no curvative effect for rickets, while a few minutes radiation per day with a bare quartz mercury arc was sufficient to cure a very bad case of rickets in chickens. The direct radiation from the sun will also cure rickets[2] but the same rays through common window glass have little, if any, value as a cure for rickets. These experiments show that the radiation suitable for this purpose is not transmitted by common window glass. The short wave-length transmission limit of window glass varies with the particular glass considered, but very little is transmitted below 3100A which places a rough long wave-length limit to the radiation that is of value for this one purpose.

When the body is exposed for a sufficient time to the proper intensity of ultraviolet radiation of wave lengths centering around 3000A a reddening of the skin appears in a few hours that persists for some time. This reddening of the skin is spoken of as an erythema. The radiations of different wave lengths are not equally effective in producing erythema.

Hausser and Vahle[3] have measured the relative effect of radiation of different wave lengths for producing erythema and have given their results in the form of a curve which is shown in Figure 1. The short

[1] Am. Jour. Physiol., *87*, p. 391; December, 1928.
[2] Am. Jour. Med. Assoc., *88*, p. 1362; May, 1927.
[3] Strahlenther, *13*, p. 41; 1922.

wave-length part of this curve is not as accurately known as that for longer wave lengths but for calculating the relative erythema produced by sources at high temperature only a slight error will be produced since there is such a small amount of enegy in these short wave-length regions.

The radiation characteristics of non black bodies are generally given by comparing them with the standard radiator, that is, the black body. It is well known that the relation between the intensity of radiation

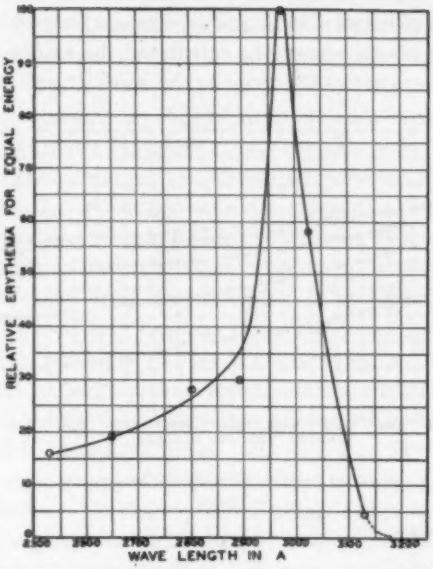

FIG. 1. *Relative effectiveness of ultraviolet radiation of different wave lengths in producing erythema.*

and the wave length can be calculated for different temperatures of the black body by means of Planck's equation. Considerable work has been done on the radiation characteristics of tungsten. Weniger and Pfund[4] made a study of the infrared radiation of tungsten, extending their work to wave lengths of about 4μ. They showed that for wave lengths longer than about 2μ the spectral emissivity e_λ could be calculated for different temperatures by means of Drude's formula

$$100 - R_\lambda = e_\lambda = 3650\sqrt{\rho/\lambda}$$

[4] Phys. Rev. *14*, p. 427; 1919.

where ρ is the specific resistance for the temperature considered. R_λ the reflectivity, and λ the wave length in μ. Making use of this relation the emissivity of heated tungsten can be calculated for longer wave lengths and for higher temperatures than they extended their measurements. Several investigations[5] have been made in the laboratory at Nela Park on the visible radiation of tungsten with the result that the distribution of energy in this part of the spectrum is quite well known. Hulburt,[6] working in the laboratory at Nela Park, made a study of the ultraviolet radiation from tungsten, extending his work to wave lengths of about 3400A and to temperatures of about 2800 deg. K. From these measurements he calculated the emissivity of tungsten for the ultraviolet part as far down as the wave length 3400A. The final

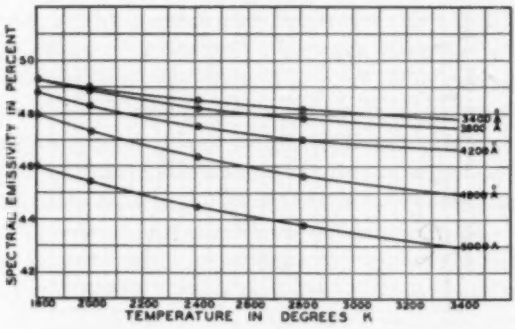

FIG. 2. *Spectral emissivity of tungsten for different temperatures and several different wave lengths.*

O—Hulburt's values.

values that he obtained, corrected to the present temperature scale of tungsten, are reproduced in Figure 2. His test points are shown and the curves have been extrapolated to 3400 deg. K. From the curves it seems that these extrapolated values cannot be in error by as much as five percent. We are interested, however, in knowing the radiation down to 3000A, and if possible, down to about 2800 or 2900A, since it is radiation of wave lengths in the neighborhood of 3000A that are supposed to be effective in promoting health and growth in plants and animals.

[5] Phys. Rev., *10*, p. 397; 1917.
Phys. Rev., *10*, p. 377; 1917.
J.O.S.A. *7*, p. 1115; 1923.
[6] Astrophys. Jour., *45*, p. 149; 1917.

The energy distribution in the visible portion of the tungsten spectrum may be calculated by making use of the color temperature of tungsten.[7] By color temperature is meant the temperature at which it is necessary to operate a black body so that the emitted light from the black body will be of the same color as that of the source under investigation. For instance, it was found that the light from a tungsten lamp operated at a temperature of 2400 deg. K. has the same color as the light from a black body at 2450 deg. K. From this the color temperature of tungsten corresponding to a true temperature of 2400 deg. K. is 2450 deg. K. Color temperature thus defined and measured refers to the visible spectrum and not to the infrared or ultraviolet parts of the spectrum.

By extended tests it has been shown[8] that when a tungsten filament and a black body are color matched they have the same spectral distribution to within about one percent throughout the visible spectrum. The spectral distribution of energy from hot tungsten varies from that of a black body at color match in such a manner that if these two radiators are set so as to have the same ratio of red brightness to blue brightness, the radiation from the tungsten in the green portion of the spectrum will be relatively brighter than that from the black body. If this same relation holds for shorter wave lengths, measured values of emissivity should be smaller than values calculated from color temperature relations. Such calculations have been made for wave length 3400A and for a color temperature of 2800 deg. K. The calculations gave 0.495 as the emissivity, while Hulburt's test value for this same temperature is 0.485. This shows that Hulburt's measured value of emissivity for this region is in general agreement with the work in the visible spectrum.

From the foregoing relations and data, the emissivity of tungsten for a temperature of 2800 deg. K and for wave lengths from 3400A to 40,000A has been calculated, and the results plotted in Figure 3. This curve has been extrapolated to 2700A. Similar curves have been plotted for other temperatures where the values for the shorter wave lengths have been obtained by cross-extrapolation where necessary. These extrapolations assume, of course, that there is no sudden change in the radiation characteristics of tungsten. Almost all metals become poor reflectors, i.e., good radiators, for short wave lengths. The curve for tungsten shows a trend in that direction. Any change probably

[7] Astrophys. Jour., *61*, p. 176; 1925.
[8] J.O.S.A., *7*, p. 1115; 1923.

would be toward a smaller value for the reflectivity rather than in the opposite direction. It is therefore thought that the values of emissivity given for the far ultraviolet are, if in error, smaller than the true values. Work is now under way in this laboratory to check these values by actual measurements in the ultraviolet region.

The spectral distribution of energy radiated by tungsten at different temperatures has been calculated for a range of wave lengths from the very short ultraviolet to the long infrared and by step-by-step integration the relative amount of energy in the various portions of the spectrum of tungsten at different temperatures has been calculated and

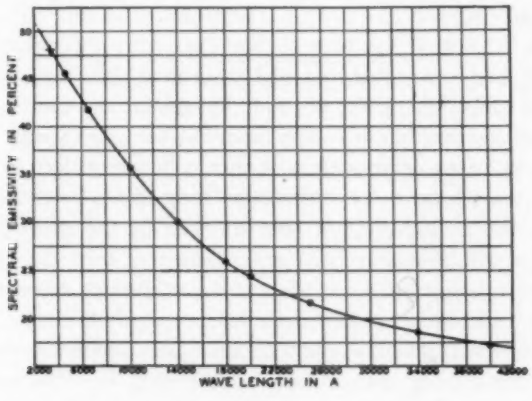

FIG. 3 *Spectral emissivity of tungsten for 2800°K.*
○—*Weniger and Pfund's values.*
●—*Worthing's values.*
+—*Hulburt's value.*

is shown in Table 1. From these relative values the amount of energy in different wave length intervals falling upon a square centimeter at a distance of one meter from different lamps was calculated and is also shown in Table 1.

The spectral distribution of the energy received from the sun at the earth's surface can be calculated from the distribution of energy in the spectrum of the sun outside of the earth's atmosphere and the transmission of the atmosphere. Abbott[9] and his co-workers of the Smithsonian Institution have published data on the sun's spectrum

[9] Smithsonian Miscellaneous Collection, *74*, p. 15; 1923. Annals Astrophysical Observatory, *2*, p. 112; 1908.

TABLE 1. *Energy in various parts of the spectrum of the sun for various air masses and for some tungsten lamps*

Region	Sun(a)							
	1 atmosphere		1.07 atmosphere (b)		1.5 atmosphere (c)		2.37 atmosphere (d)	
	Percent	Milliwatts	Percent	Milliwatts	Percent	Milliwatts	Percent	Milliwatts
Below 3100A	0.022	0.024	0.019	0.020	0.0063	0.0061	0.00051	0.00041
Below 3250A	0.24	0.26	0.21	0.22	0.11	0.11	0.029	0.023
Below 3500A	1.1	1.17	1.0	1.07	0.64	0.63	0.27	0.22
Below 4000A	3.9	4.22	3.8	3.97	2.8	2.70	1.62	1.3
4000–7600A	43.7	46.8	43.9	45.8	42.1	40.4	39.1	31.4
Total		107.0		104.5		96.1		80.2
	Tungsten filament lmps (e)						Tungsten arc (f)	
	500-watt, 1000 hr.		500-watt, 300 hrs.		900-watt, movie		Percent	Milliwatts
	Percent	Milliwatts	Percent	Milliwatts	Percent	Milliwatts		
Below 3100A	0.017	0.00068	0.022	0.00088	0.046	0.0042	0.098	0.0112
Below 3250A	0.032	0.0013	0.042	0.0017	0.083	0.0076	0.26	0.030
Below 3500A	0.08	0.003	0.10	0.004	0.18	0.017	0.52	0.059
Below 4000A	0.31	0.012	0.38	0.015	0.63	0.058	1.53	0.17
4000–7600A	15.7	0.63	16.7	0.67	19.4	1.8	27.0	3.1
Total		4.0		4.0		9.2		11.4

(a). The rays from the sun are assumed to fall upon a surface perpendicular to the direction of propagation.
(b). For June 21 at latitude of Cleveland (O).
(c). For Feb. 15 at latitude of Cleveland.
(d). For Dec. 21 at latitude of Cleveland.
(e). The energy from the tungsten lamp is the amount falling on a square centimeter at a distance of one meter from the lamp.
(f). Calculated for one square centimeter of molten tungsten.

and on the transmission of the atmosphere. These values supplemented for very short wave lengths by the values published by Fabry and Buisson[10] have been used as a basis for determining the energy of different wave lengths that reach the earth's surface from the sun. In Fig. 4 is shown the transmission of the atmosphere for the short wave-length region that has been used in this calculation. These values were arrived at as a result of a comparison of all the available data. The actual values of the sun's spectral intensity in the ultraviolet region that were used were obtained by following the findings of Abbott,[11] who made an extended analysis concerning the distribution of energy in the ultraviolet portion of the sun's spectrum. Fabry and Buisson

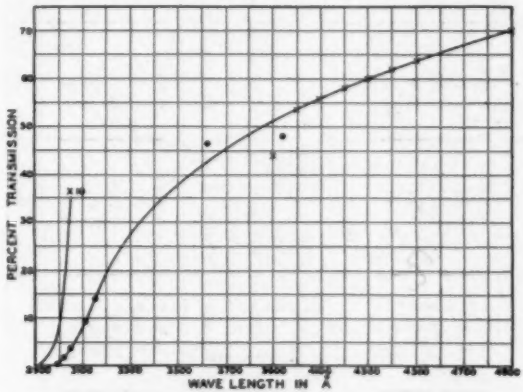

FIG. 4. *Spectral transmission of the atmosphere for short wave lengths.*
●—*Fabry and Buisson's values.*
×—*Abbot's values.*
Curve marked × 10 same as main curve but plotted with the transmission multiplied by 10.

in their determination of the radiation in the ultraviolet portion of the sun's spectrum showed that the amount of energy that reaches the earth's surface for wave length 2900A is only one-millionth that for wave length 3140A; which shows that the energy for wave length 2900A is extremely small because even at 3140A the intensity is small. The fact that the energy that reaches the earth's surface is practically zero for 2900A and shorter wave lengths is thought to be due to absorption

[10] Comptes Rendues, *175*, p. 156; 1922.
Astrophys. Jour., *54*, p. 297; 1921.
[11] Sonderdruch aus "Gerlands Beitrage zur Geophysik", XVI, p. 344; 1927.

in the upper atmosphere, probably by ozone in that region, since it is quite well known that the sun sends out considerable energy at these short wave lengths. From the distribution of energy in the sun's spectrum that reaches the earth's surface, the actual percentages in the different wave-length regions were obtained by step-by-step integration. Making use of Abbott's[12] solar constant, it is possible to express these percentages in the various wave-length intervals of the sun's spectrum in watt's per square centimeter. The results for different wave-length intervals given in Table 1 are for the direct rays of the sun and not for any radiation that may reach a square centimeter of surface due to scattering. These values are for the average distance of the sun from the earth. If the variation of the distance of the sun from the earth is considered, it will give a variation of about ± 3 percent in the energy. In this calculation the absorption of the sun's energy by water vapor has also been neglected. If this were taken into account it would increase these values by a few percent.

One check of the amount of energy in the solar spectrum for this region was obtained by making use of some results in a paper by Holladay.[13] In this paper there was presented a very convenient method of calculating the percentage of energy in the different spectral regions of a black body. The distribution of energy in the solar spectrum outside the earth's atmosphere corresponds very nearly to that of a black body[10] at 6000 deg. K. According to Holladay's results, there is about 2.7 percent of the energy of a black body at 6000 deg. K in the region between 2900 and 3250A. The transmission for the curve given in Fig. 4 for the energy of a black body in this region at this temperature amounts to about 8 percent. This indicates that there is about 0.22 percent of the energy of the sun between wave lengths 2900 and 3250A that reaches the earth's surface, which is in very good agreement with the foregoing results.

Coblentz[14] gave some data on the ultraviolet energy of the sun as measured at Washington. These measurements showed that on clear days about 2 percent of the solar energy that reaches the earth's surface is of wave lengths shorter than 3900A. This is also in good agreement with the preceding results obtained by calculation.

Kimball gave the results of some calculations on the distribution of energy in the solar spectrum in a paper presented before the International

[12] Annals of Astrophysical Observatory, *4*, p. 192; 1922.
[13] J.O.S.A., & R.S.I., *17*, p. 329; 1928.
[14] Bulletin of Bureau of Standards, *16*, p. 240; 1920.

Illumination Congress at the meeting at Saranac in September, 1928. His conclusions agree well with the values given in this article, where these overlap.

Prof. Winchester[15] has been studying the ultraviolet wave-length limits of the sun's energy at New Brunswick (N. J.) for a number of years. In Figure 5 is shown his curve for these limits for different periods of the year. The values for the equinoxes for his wave-length limits were taken from the foregoing calculation of the energy reaching the earth's surface, assuming that this was the minimum amount of energy necessary to affect the photographic plate. The wave lengths for this same energy for four other values were plotted on his curve. As is shown in the curve, the agreement is good.

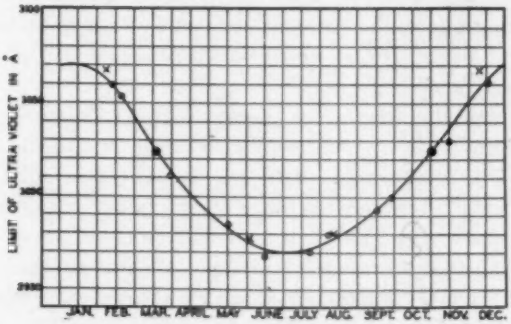

FIG. 5. *Prof. Winchester's limit of blackening on photographic plates for north sky at New Brunswick (N. J.) for different seasons of the year. The vertical line above each month denotes its beginning.*

○—*Winchester's values.*

●—*Two points from calculations in this article made to fit the curve.*

×—*Other points plotted from data in this article.*

The energy from the sun when it is directly overhead is more than people who are not accustomed to this intensity can stand; and even people living in such regions do not spend much time directly exposed to the sun's rays. It is to be noted that the sun is never directly overhead except for a region 23 1/2 deg. north or south of the equator, and even in this region only at noon. This amount of ultraviolet energy is probably very greatly in excess of what is needed for the promotion of health and growth in human beings. It is almost certain that a very short exposure to such radiation will result in very violent sunburn.

[15] Unpublished data furnished by Prof. George Winchester. See also Riddle, J.O.S.A. & R.S.I., *18*, p. 442, June, 1929.

Different people have often reported that they are very badly sunburned by exposure of their arms and face while playing tennis in Cleveland after 4:30 P.M. (sun time). The length of the path of the sun's rays through the atmosphere varies as the secant of the angle of the sun's zenith distance. Making use of this relation and the transmission for one atmosphere the transmission of the earth's atmosphere for different wave lengths was calculated for the following air masses: 1.07, 1.2, 1.5, 2.0, 2.37, and 3.0. From these transmissions and the spectral intensity of the sun's energy outside the atmosphere, the intensity of the sun's energy reaching the earth's surface was calculated

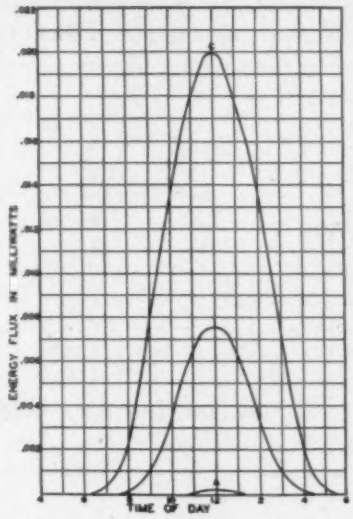

FIG. 6. *Ultraviolet energy below 3100A from the sun falling on a square centimeter of the earth's surface per second for latitude of Cleveland for different times of the day and different seasons.*

Curve A—winter solstice.
B—spring and fall equinoxes.
C—summer solstice.

for the different air masses named, and the amount in the different wave-length intervals calculated.

By plotting a curve showing the relation between air mass and intensity of solar radiation for different wave-length regions the amount of energy for the wave length regions reaching the earth's surface at different times of day and at different seasons of the year can be obtained. The curves in Fig. 6 show the intensity of the energy in the

sun's spectrum below 3100A for four periods of the year: winter and summer solstices, and spring and fall equinoxes. These curves show the very great variation of the ultraviolet radiation for different times of the day and different seasons of the year.

In Fig. 7 is shown the variation of the ultraviolet energy below 3100A at noon for different times of the year for the latitude of Cleveland.

In Fig. 8 is shown the relative amount of energy in the ultraviolet region radiated per second by a 900 Watt movie lamp and the sun. These values are for the amount of energy per second falling on a square centimeter at a distance of one meter from the lamp in a direction normal to the plane of the filament and the amount falling on one square centimeter of horizontal surface from the sun directly overhead.

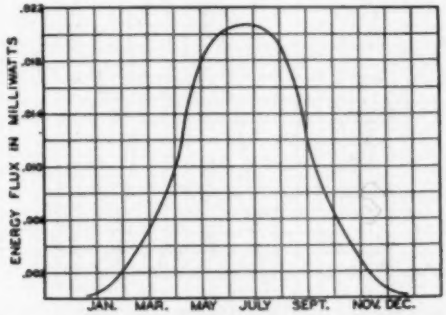

FIG. 7. *Ultraviolet energy below 3100A from the sun falling upon a square centimeter of the earth's surface per second for the latitude of Cleveland for different times of the year for 12 o'clock noon. The vertical line above each month denotes its beginning.*

The data in Table 1 show that the amount of the sun's energy in the region below wave length 3100A that reaches the earth is not very much greater than from the 900-Watt movie lamp at a distance of one meter, even for the best conditions possible for the sun. If there is any very great amount, of dust or smoke, the ultraviolet radiation from the sun is very greatly reduced, if not wholly absorbed.[16] It is very doubtful if any of the ultraviolet energy in this region is received from the sun in winter at or near a great city where the dust and smoke are quite thick.

In Table 2 is given the ultraviolet energy in the region below 3100A and the erythema producing power of the sun and some tungsten fila-

[16] Bundesen, Lemon, et al., Jour. Amer. Med. Assoc., *89*, p. 187; 1927.

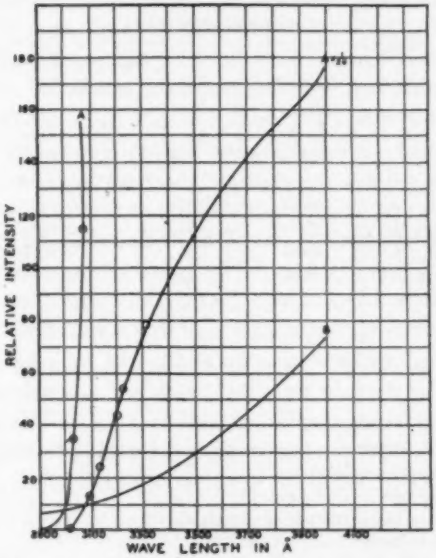

FIG. 8. *Energy distribution in the ultraviolet region. Values as plotted show the amount of energy per second falling on a square centimeter of the earth's surface for the sun directly overhead, and the energy per square centimeter per second from the tungsten filament at a distance of one meter.*

Curve A—energy from the sun.

B—energy from the 900-watt tungsten movie lamp perpendicular to plane of filament.

○—unpublished values of Brian O'Brien.

Curve marked A × 1/20 same as curve A but plotted with the energy multiplied by 1/20.

ments.[17] For comparison in Columns 2 and 3 the illumination and total energy received per square centimeter from each source is included. The figure for tungsten filaments have been calculated without allowing for the absorption of the bulb. For an ordinary glass bulb, of course, the very short wave-length ultraviolet energy is wholly absorbed and thus eliminated. If a quartz bulb is used, about 10 percent must be allowed for the factor, i.e., the figures given would have to be reduced by about 10 percent. The transmission of the quartz substitute glass varies from about 30 to 80 percent for this region.

From the data in Table 2 it can be seen that it is possible to obtain per square centimeter an amount of ultraviolet energy from the tungsten filaments that is of the same order as that obtained from the sun.

[17] Compare Stockbarger, et al., J.O.S.A., & R.S.I., *18*, p. 53; Jan. 1929.

TABLE 2. *Total energy, illumination, and ultraviolet energy from the sun and some tungsten filament lamps*

Source	Illumination (Foot-candles)	Energy flux per cm.² (milliwatts)		Relative erythema
		Total	Below 3100A	
Sun (a) 1 atm.	8540	107	0.024	100.
1.05 atm. (b)	7900	97.6	0.021	88.
1.5 atm. (c)	4940	64.0	0.0041	26.
2.37 atm. (d)	2415	33.9	0.00017	2.6
500-watt tungsten (e) lamp, 1000 hr. life	71	4.00	0.00068	2.9
500-watt tungsten lamp, 300 hr. life	87	4.00	0.00088	3.8
900-watt movie (e)	292	9.2	0.0042	6.4
Tungsten arc (f)	544	11.4	0.0112	25.

(a)The energy and illumination from the sun are corrected for the sun's distance from the zenith, i.e., are for a horizontal surface.

(b) June 21 for latitude of Cleveland (O).

(c) About February 15 for latitude of Cleveland (O).

(d) December 21 for latitude of Cleveland (O).

(e) The values for the tungsten filament lamps and arc for a cm.² at a distance of one meter from the lamp.

(f) Calculated for one cm² of molten tungsten.

If the comparison is made on the basis of equal energy or equal illumination, it is much more favorable for the tungsten filaments. For the condition of equal total energy per square centimeter more ultraviolet energy below 3100A will be received from the 900-Watt movie lamp filament at a distance of one meter than from the sun in winter; while for the sun directly overhead the movie lamp filament will show about as much ultraviolet energy in this region as the sun. The ultraviolet energy below 3100A from the 500-Watt tungsten filament lamp operated at 300 hours' life compares very favorably with that from the sun on the basis of equal total energy. By the use of reflectors with the tungsten lamps, the amount of energy in the different wave-length regions falling on a square centimeter as considered could be very greatly increased.

The relative erythema produced by different sources can be determined from their energy distribution and the relative erythema producing power for radiation of different wave lengths given by the curve in Fig. 1. This is done by forming the product of the intensity of the energy at different wave lengths and the erythema producing power for

the corresponding wave length and then integrating the result by the step-by-step method. Such products have been formed and integrated for both the black body and tungsten at different temperatures and the results for the relative erythema produced by these sources for different temperatures is shown in Fig. 9. The results for tungsten are for a 500-Watt lamp of approximately uniform spherical distribution with no allowance for the absorption of the bulb and are for one square centimeter at a distance of one meter from the lamp. The values for the black body are also for a radiation of 500 Watts and for one square centimeter at a distance of one meter from the black body and in a direction perpendicular to its plane. The black body consists of the usual circular opening that radiates toward one side only. If everything else were equal this would give four times as much energy for the black body as for the tungsten at the same temperature and the same distance. Due to the selectivity of tungsten for shorter wave lengths this differential is reduced as shown by the curves in Fig. 9. The relative erythema that is produced by the sun for different elevations was also

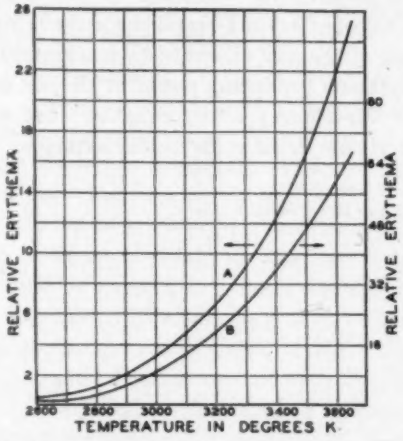

FIG. 9. *Relative erythema produced by the black body (curve B) and tungsten filament (curve A) at different temperatures.*

calculated and is shown in Table 2. The results are for a square centimeter that is perpendicular to the suns rays. All the values of relative erythema given in the paper for different sources are in terms of that produced by the sun directly overhead with a clear atmosphere. This value is taken as 100.

Fig. 10 shows the relative erythema that will be produced at the latitude of Cleveland for different times of the year. These results

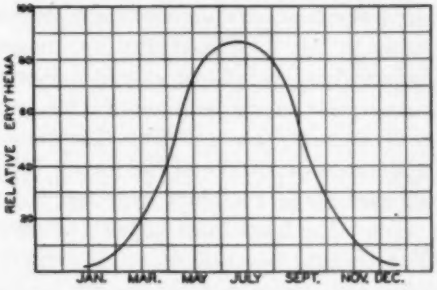

FIG. 10. *Relative erythema that will be produced by the sun on clear days at the latitude of Cleveland for different times of the year. The vertical line above each month denotes its beginning.*

show that the effect of the sun for producing erythema at this latitude varies by a factor of about 35 between June 21 and December 21. Plotting a curve showing the variation of the erythema producing power of the sun for the 21st of December for the latitude of Cleveland for different times of the day and then by integration it can be shown that the total erythema producing power of the sun on December 21st at the latitude of Cleveland is about equal to what would be received from the sun directly overhead in about four minutes.

LAMP DEVELOPMENT LABORATORY,
INCANDESCENT LAMP DEPT. OF G. E. CO.

THE WAVE LENGTHS OF COMPLEMENTARY HUES

By I. H. Godlove

The experimental data on the wave lengths of the spectral stimuli which evoke complementary hues has been summarized in Helmholtz' treatise[1] in tabular form, and graphically by Priest.[2] The latter gives also a graph of average data for the seven observers, and an equation to fit the curve, the coordinates having been put in the form of complementary frequencies. Grünberg[3] gives an equation in terms of wave lengths. These equations are

$$\text{Priest} \quad (530-f)\ (f'-608)=220$$
$$\text{Grünberg} \quad (\lambda-559)\ (498-\lambda')=424$$

where λ is the wave length in millimicrons, λ' is the wave length of the complementary hue; and $f=299796/\lambda$; $f'=299796/\lambda'$ where f and f' are in vibrations per trillionth of a second (fresnels). The curves of Priest's and Grünberg's averages are hyperbolae, and the curves for even the individual observers "nearly resemble hyperbolae."[4] For this reason the hyerpbolic form will be assumed in the following discussion, although there is no theoretical necessity for it.

Since it is possible, by a simple change of variables, to plot such data as straight lines, it seemed that thereby a simpler graphical picture of the variations in the color vision of various observers could be obtained, at least insofar as complementarism is concerned. If the "white points" of the observers lie on the same straight line in the "color triangle," the straight lines of the present work should pass through or near a common point, so that the differences between such observers would be reduced to a single variable (as the slope) on the graph. This has been realized in the case of 14 of the 15 observers considered in the present paper, as shown in Figs. 1 and 3; and the fifteenth one had already been shown to have an "abnormal" white point by Sinden.[5] The method described in the next paragraph involves the use of a reference point (the same for all observers), and from the lines obtained fixing a common point. The common point here obtained may be considered the norm from this point of view, and may in turn be used as the reference point for new observers. If any new

[1] Helmholtz, *Physiological Optics*, Engl. Transl. II, 126, 128; 1924.
[2] Priest, J. Opt. Soc. Amer., *4*, 402–4; 1920; *5*, 513–4; 1921.
[3] Grünberg, Sitzb. Akad. Wien, II A, *113*, 627–36; 1904.
[4] Parsons, *Introduction to the Study of Color Vision*, p. 35; 1915.
[5] Sinden, J. Opt. Soc. Amer. *7*, 1123–53; 1923.

observer is "normal," *his complete range of complementary wave length pairs may be obtained from the observation of two pairs.*

The method is as follows. Choose any convenient reference point (λ_0, λ_0') on (or sufficiently near) the smooth curve for the observer, λ_0 and λ_0' being a pair of complementary wave lengths. Then for each experimental pair find the differences between the coordinates of the pair and the respective coordinates of the reference point. Plot the ratio of differences $\lambda-\lambda_0/\lambda'-\lambda_0'$ as ordinates and the differences $\lambda-\lambda_0$ as abscissa. Then pass the "best" straight line through the plotted points. This line may be described by the equation

$$\frac{\lambda - \lambda_0}{\lambda' - \lambda_0'} = S(\lambda - \lambda_0) + I \tag{1}$$

where S is the slope of the line and I its intercept on the ratio-of-differences axis. If we simplify this equation, arrange it in terms of λ and λ' and their product, and make the following substitutions for the constant quantities, viz.,

$$a = -\left(\frac{S\lambda_0 - I}{S}\right) = \frac{I}{S} - \lambda_0 \tag{2}$$

$$b = -\left(\frac{S\lambda_0' + 1}{S}\right) = -\left(\frac{1}{S} + \lambda_0'\right) \tag{3}$$

$$k = -\left(\frac{\lambda_0 + S\lambda_0\lambda_0' - I\lambda_0'}{S}\right) \tag{4}$$

we obtain finally

$$\lambda\lambda' + b\lambda + a\lambda' = k \tag{5}$$

Adding ab to each side, and substituting

$$c = k + ab \tag{6}$$

we obtain

$$\lambda\lambda' + b\lambda + a\lambda' + ab = c \tag{7}$$

which may be written,

$$(\lambda + a)(\lambda' + b) = c \tag{8}$$

which is the hyperbolic form of Priest's and Grünberg's equations. By measuring S and I on the graph and applying equations 2, 3, 4 and 6, the data in Table 1 has been obtained. For Fig. 1 and this table, a

reference point not entirely constant was used; instead for convenience actual observed points each near the point (583, 481) were used for the respective observers. It is seen that the lines of Fig. 1 do not intersect

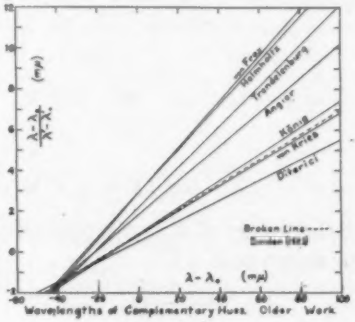

FIG. 1

as exactly in a common point as do the lines of Fig. 3, where exactly the same reference point was used throughout. Nevertheless the common point may be located exactly as follows.

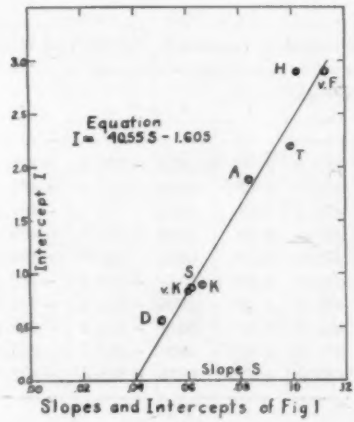

FIG. 2

This results from the fact that, insofar as the common intersection is exact, I and S are not independent of each other. Fig. 2 is a graph of I as a function of S for the data of Fig. 1; and the line may be represented by

$$I = sS + i \tag{9}$$

where s is its slope and i its intercept on the I axis. If we now write Eq. 1 in the form

$$w = \frac{x}{y} = Sx + I \tag{10}$$

where

$$x = \lambda - \lambda_0 \quad \text{or} \quad \lambda = x + \lambda_0 \tag{11}$$

and

$$y = \lambda' - \lambda_0' \quad \text{or} \quad \lambda' = y + \lambda_0' \tag{12}$$

and if w_c and x_c are the values of w and x respectively at the common point, then it may be shown that

$$s = -x_c \tag{13}$$

and

$$i = w_c. \tag{14}$$

For the data of Table 1 we thus obtain for this point the coordinates $(-40.55, -1.605)$.

TABLE 1. *Constants of Equation 8. Earlier Work (Helmholtz).*

Observer	Reference Point λ_0	λ_0'	I	S	a	b	c	c Indirect
Helmholtz	585.3	485.4	2.89	0.1079	−558.5	−494.7	−248.2	−244.6
v. Kries	579.7	478.7	0.83	.0602	−565.9	−495.3	−229.0	−235.1
v. Frey	586.7	478.7	2.89	.1125	−561.0	−487.6	−228.3	−231.6
König	582.6	483.6	0.89	.0654	−569.0	−498.9	−208.1	−214.4
Diterici	578.0	476.6	0.55	.0497	−566.9	−496.7	−222.7	−217.8
Angier	583.3	480.6	1.88	.0834	−560.8	−482.8	−270.3	−258.9
Trendelenburg	583.5	482.8	2.19	.0994	−561.5	−492.9	−221.6	−231.7
(Sinden)	580.0	482.5	0.86	.0613	−566.0	−498.8	−228.9	−226.8
Av. of all	582.4	481.1	1.622	.0800	−563.7	−493.5	−232.1	−232.6
Av. of first 7	582.7	480.9	1.731	.0826	−563.4	−492.7	−232.6	−233.4

This common point may now in turn be used as the reference point for new observers; and from two pairs of complementaries the lines may be drawn, and the complete range of wave lengths determined for the new observers. This has been done for the data of the eight observers reported by Sinden,[5] and the results are shown in Table 2 and Figs. 3 and 4. One of the observers (Sinden) of the second set is also shown for comparison with the first set (and by the broken line

in Fig. 1). Sinden states explicitly two pairs for each observer, as well as the wave lengths of hues complementary to the red and the violet regions of invariant hue. For the computations of Table 2, the red

TABLE 2. *Constants of equation 8. Sinden's observers.*

Observer	Using in each case		$\lambda_0 = 541.85$ and $\lambda_0' = 506.35$				
	I	S	a	b	c	c Indirect	Av. Dev. (Calc. —Obs.)
A H P	3.55	−.1408	−567.1	−499.2	−179.1	−181.4	±0.55
A E	7.90	−.2095	−579.6	−501.6	−180.0	−179.4	±0.78
R H S	2.82	−.1164	−566.1	−497.8	−208.1	−218.2	±0.48
H T W	10.55	−.3567	−571.4	−503.6	− 82.9	−118.3	±2.25
B K	2.22	−.0976	−564.6	−496.1	−233.1	−247.0	±1.15
H N E	1.34	−.0694	−561.2	−491.9	−278.2	−275.6	±1.00
P F G	2.38	−.1060	−564.3	−496.9	−211.8	−230.3	±1.25
R	3.90	−.1491	−568.0	−499.6	−175.4	−167.7	±0.18
Average of all	4.33	−.1557	−567.8	−498.3	−193.6	−202.2	±0.96
Av., excluding A E and H T W	2.70	−.1132	−565.2	−496.9	−214.3	−220.0	±0.77

was arbitrarily taken as 680 mμ, and the violet as 420 mμ. It will be noted that 7 of the 8 observers give lines in Fig. 3 passing accurately through a common point (32.25, −0.95). The line for the average of

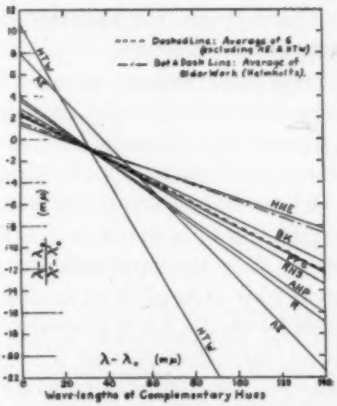

FIG. 3

the Helmholtz group also passes quite close to this point. The line for *AE* does not. That this observer is abnormal in his "white point" was

pointed out by Sinden. Another line (for H T W) passes through the common point but has an abnormally large slope. This is connected with the fact that the white point for this observer is in an extreme position along the line of the color triangle through the white points of the various observers. Similarly, it may be noted by a glance at Fig. 1 that König, von Kries, Diterici and Sinden all have similar white points, while von Frey's, for example, is quite different. It is impossible to say whether in the case of the earlier observers, these differences are due to the use of different neutral lights with which

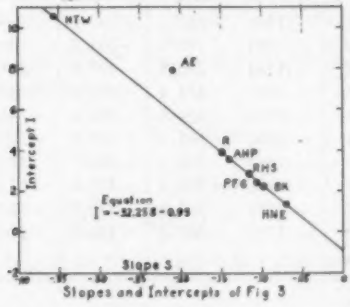

FIG. 4

to match the complementary pairs, or to differences between the visions of the observers; but, in the case of Sinden's observers, since the same comparison light was used by all, the variations must be ascribed to different visual systems of the observers.

It should be noted that since different methods of averaging may be applied, the mean constants obtained by these methods will not check exactly. For example, an a and b computed from average I's and S's will not be exactly the same as an a and b obtained from an average of several a's and b's each computed separately from the corresponding I and S. It was thought best to compute an a and b for each observer and then average them, since they are sufficiently constant that the method of averaging can have little effect on the final average. The same is true to a lesser extent for the c's, but a given error in c has in general a relatively smaller effect than that of a or b on the computed complementary wave lengths. c is computed from the following equation obtained by combining equations 2, 3, 4 and 6:

$$c = -\frac{I}{S^2}. \tag{15}$$

The "indirect" c in the last column of the tables is obtained by substituting each of the experimental complementary values of λ and λ' along with the respective a and b into equation 8, and averaging the results.

The common point for the lines of Table 1, interpreted to wave lengths by means of equations 11 and 12, is the point (541.85, 506.35) which is used as the reference point of the second group; while that for Sinden's observers (excluding AE) is the point (574.1, 472.4). The former point is on the second branch of the hyperbola which corresponds to the gap for which there are no actual physical pairs of wave lengths stimulating a complementary pair of hues; the latter point, however, lies on the branch of the curve describing the actual complementarism.

For comparison with Priest's equation we may substitute the frequency-wave-length relation into our average equation for the Helmholtz group. We thus obtain

$$(532.6 - f)(f' - 609.0) = 271.0. \qquad (16)$$

For the range of f from 500 to 525 fresnels, this equation gives values of f' differing by 4 or 5 fresnels from Priest's.

The average equation for the second group gives complementaries for the region 680 to 585 mμ differing from the corresponding ones of the first group by about 4 mμ; and for the region 430 to 475 mμ by about 1 mμ. The two curves intersect, *i.e.*, agree exactly, at the point (572.2, 466.3). Priest's average data lies for the most part between the two. The agreement of all three is excellent from 430 to 470 mμ. Another set of complementaries may be obtained from the trichromatic data of the color triangle for the Abbott-Priest noon sun used in the Bureau of Standards Colorimetry Section. This gives a curve lying also between our two averages in the region 680 to 585 mμ, but differing by 3 to 5 mμ at the violet end.

SUMMARY

A method of rectifying the hyperbola describing the wave lengths of complementary hues has been described and applied to the data of 15 observers grouped into two groups. It is shown that the results of these observers may be used to define a normal reference point in such a way that the whole range of complementary wave lengths may be computed from two observed pairs. The straight lines through the two corresponding points meet in a common point, and the slope of the

lines gives at a glance the characteristic differences between observers so defined as normal. An observer having a white point, abnormal in the sense of not falling on the line of the color triangle defined by 14 of the 15 observers here considered, will give a line not passing through the common point. The method gives a simple accurate way of finding the constants of the equations of the hyperbolae for individuals, and a simple way of finding an average.

MUNSELL COLER COMPANY, INC.,
10 E. FRANKLIN STREET,
BALTIMORE, MARYLAND.

OPTICAL SOCIETY OF AMERICA

Statement of Cash Receipts and Disbursements

For the Period from January 1, 1929 to December 31, 1929

Cash on Hand, January 1, 1929		$5,574.97
Receipts:		
Membership Dues	$4,119.35	
Subscriptions to the Support of the Society	3,000.00	
Interest on Bank Balances	279.58	
Interest on Bonds	433.75	
Proceeds from Sale of Advertising	160.00	
Share of the Society in Journal Subscriptions	197.40	
Share of the Society from Advertisements in Journal	425.00	
Sale of Volumes I–V of Journal (Property of Society)	202.60	
Sale of Colorimetry Reports	5.95	
TOTAL RECEIPTS		8,823.63
Total Receipts plus Cash on Hand, January 1, 1929		$14,398.60
Disbursements:		
To Journal of the Optical Society of America and Review of Scientific Instruments, as per Contract with Association of Scientific Apparatus Makers of U. S.	$4,800.00	
Journal Subscriptions (180 @ $3.00 each)	540.00	
Printing and Stationery	669.13	
Postage and Telegraph in Connection with Business of Society	142.92	
Postage on back numbers of Journal	8.82	
Traveling on Business of Society	85.40	
Clerical Expenses	93.11	
Fire Insurance Vols. I–V of Journal	8.76	
Frederick Ives Medal and Case	18.26	
Labor and Transportation on Addressograph	75.43	
		6,441.83
Purchase of Liberty Bonds for Investment	2,535.16	

Accrued Interest of Liberty Bonds for Investment	22.43	
		2,557.59
		$8,999.42
Cash on Hand, December 31, 1929 (including Life Membership Fund $500.)		5,399.18
Total Disbursements plus Cash on Hand, Dec. 31, 1929		$14,398.60

Investments:

$5000 Atchison, Topeka and Santa Fe Gen. 4's.

$7500 Fourth Liberty 4–1/4's.

$ 500 Fourth Liberty 4–1/4's (Frederick Ives Medal Endowment Fund)

I certify that the above report is correct to the best of my knowledge and belief.

ADOLPH LOMB, *Treasurer*

COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK
DEPARTMENT OF PHYSICS

February 8, 1930

Adolph Lomb, Esq.
Treasurer, Optical Society of America:
Dear Mr. Lomb,

In accordance with your request we have audited the treasurer's accounts of the Optical Society of America for the twelve months ending December 31, 1929; and we hereby certify that the treasurer's official report of receipts, disbursements and investments for the year 1929 is correct in every detail, as shown by bank statements, vouchers, etc.

Yours truly,

JAMES P. C. SOUTHALL
C. L. TRELEAVEN

JOURNAL OF THE OPTICAL SOCIETY OF AMERICA AND REVIEW OF SCIENTIFIC INSTRUMENTS

Statement of Cash Receipts and Disbursements

For the Period from January 1, 1929 to December 31, 1929

Cash on Hand, January 1, 1929		$8,160.51
Receipts:		
In account with the Optical Society of America (12 issues at $400.00 each)		4,800.00
In account with Scientific Apparatus Makers of America Advertising		7,911.99
Miscellaneous Receipts		
Subscriptions	$3,535.18	
Sale of back numbers	564.28	
Sale of copies of March 1929 issue	206.20	
Reimbursements	149.65	
On account of Helmholtz Sale	27.00	
Interest on Deposits in (1929)		
Ithaca Trust Company	121.20	
Ithaca Savings Bank	121.20	
		4,724.71
Total		$25,597.21
Disbursements:		
Printer	$8,336.71	
Salaries and Office Expenses	4,937.29	
Miscellaneous	462.34	
		$13,736.34
Cash on hand, December 31, 1929		5,618.47
Amount in interest accounts		6,000.00
Interest received on accounts (1929)		242.40
Total		$25,597.21

The above is a correct statement of accounts to the best of my knowledge and belief.

(Signed) F. K. Richtmyer, Business Manager

We, the undersigned, duly appointed to audit the account of the Journal of the Optical Society of America and Review of Scientific Instruments at the close of business, December 31, 1929, hereby certify that we have examined the statement of receipts, and disbursements, the vouchers, check book and bank book and find them to agree with each other. We, therefore, certify that the above statement is a true statement of the accounts of the Journal of the Optical Society of America and Review of Scientific Instruments as of December 31, 1929.

(Signed) Harley Howe
Carleton C. Murdock
Auditors

Diagnostica e Technica di Laboratorio. (Diagnosis and Laboratory Technique.) Published monthly Piazza S. Domenico Maggiore No. 9, Naples, Italy. 150 lire annually.

The object of this new journal is to present in concise form the current developments in the field of scientific diagnosis throughout the world. Only short papers of a practical character are published. Original papers are followed by a short summary in French, English and German. A considerable part of the journal is devoted to abstracts of the literature on this subject.

Judging from the first issue most of the material relates to biochemical and biological laboratory tests. There is nothing on x-ray diagnosis.

G. FAILLA

The Kinematical Design of Coupling in Instrument Mechanisms. By A. F. C. Pollard, A. R. C. S., A. M. I. E. E., Professor of Instrument Design at the Imperial College of Science and Technology. 64 pages. Adam Hilger Ltd., London, Eng. Price 4s. 6d.

This book is an application to instrument design of the law of kinematics that six constraints are necessary to fix a rigid body. The author defines the terms used in the book, states the principles of kinematical design, and cites examples of the application of these principles to the design of instruments recently built by Adam Hilger Ltd., W. G. Pye and Co., Cambridge Instrument Co., E. Leitz, and others. Machine tool design of instruments depends too largely on the machines used in construction and on past experience, too little on kinematical principles, whereas kinematical design places first importance on these principles. By the latter method the forces on various members of an instrument are determined, the proper number of constraints applied, the materials decided upon, the so-called point contacts on the guiding surfaces expanded to withstand the pressures involved, and the allowable fits or clearances determined. This results in a stable instrument, in a reduction of variance in the instrument, in simpler and more economical manufacture because the workmanship is expended in the production of accurate geometrical forms rather than accurate fits, and permits the introduction of small adjustments which are often of great importance. Though this book is not intended as a complete text, it adequately sets forth the principles of the subject. It should be of great value to those who construct, repair, or use precision instruments.

Since reading the book this reviewer has applied these principles to the design of a comparator and an adjustable lamp support for the recording galvanometer of a microphotometer. A study of some of our present apparatus indicates that changes indicated by kinematical design would be very desirable. Accurate steel balls, which are easly and cheaply procurable, are widely used in these designs.

C. W. GARTLEIN